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ORIGINAL ARTICLE

Levitin–Polyak well-posedness by perturbations for systems of set-valued vector quasi-equilibrium problems

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Abstract This paper is devoted to the Levitin–Polyak well-posedness by perturbations for a class of general systems of set-valued vector quasi-equilibrium problems (SSVQEP) in Hausdorff topological vector spaces. Existence of solution for the system of set-valued vector quasi-equilibrium problem with respect to a parameter (PSSVQEP) and its dual problem are established. Some sufficient and necessary conditions for the Levitin–Polyak well-posedness by perturbations are derived by the method of continuous selection. We also explore the relationships among these Levitin–Polyak well-posedness by perturbations, the existence and uniqueness of solution to (SSVQEP). By virtue of the nonlinear scalarization technique, a parametric gap function g for (PSSVQEP) is introduced, which is distinct from that of Peng (J Glob Optim 52:779–795, 2012). The continuity of the parametric gap function g is proved. Finally, the relations between these Levitin–Polyak well-posedness by perturbations of (SSVQEP) and that of a corresponding minimization problem with functional constraints are also established under quite mild assumptions.

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1 Introduction

Well-posedness plays an important role in the stability analysis and numerical methods for optimization theory and applications. Since any algorithm can generate only an approximating solution sequence which is meaningful only if the problem is well-posed under consideration. Well-posedness for minimization problems [for short, (MP)] was first introduced by Levitin and Polyak (1966) and Tykhonov (1966), respectively. These are called Levitin–Polyak and Tykhonov well-posedness, respectively. The well-posedness of (MP) implies the existence and uniqueness of solutions of (MP), and each approximating solution sequence has a subsequence which converges strongly to a solution. The study of Levitin–Polyak type well-posedness for scalar convex optimization probiems with functional constraints was initiated by Konsulova and Revalski (1994). In 1981, Lucchetti and Patrone (1981) introduced the well-posedness for variational inequalities, which is a generalization of the Tykhonov well-posedness of (MP). Lignola and Morgan (2000) also introduced another well-posedness for variational inequalities, which is distinct from that in Lucchetti and Patrone (1981). Since then, many authors investigated the well-posedness and well-posedness in the generalized sense for optimization, variational inequalities and equilibrium problems (see, e.g., Ansari et al. 2000; Ceng et al. 2008; Chen and Wan 2011; Chen et al. 2011, 2012; Fang et al. 2008; Furi and Vignoli 1970; Giannessi 1998, 2000; Hu et al. 2010b; Lalitha and Bhatia 2010; Lignola and Morgan 2001; Li and He 2005 and the references therein). Recently, Huang and Yang (2006, 2007) and Huang et al. (2009) studied the Levitin–Polyak type well-posedness for generalized variational inequality problems with functional constraints as well as an abstract set constraint. They also introduced several types of generalized Levitin–Polyak well-posednesses and gave various criteria and characterizations for these types of well-posednesses in Huang and Yang (2007), Huang et al. (2009). Further, Jiang et al. (2009) discussed the Levitin–Polyak well-posedness of generalized quasivariational inequalities with explicit constraints, introduced four types of Levitin–Polyak well-posednesses and gave various criteria and characterizations for these types of well-posednesses. Huang and Yang (2010) and Xu et al. (2008) extended the results of Huang and Yang (2007), Huang et al. (2009), Jiang et al. (2009) to (generalized) vector variational inequality problems with abstract set and functional constraints. In Hu et al. (2010a), obtained the Levitin–Polyak well-posedness of variational inequalities and optimization problems with variational inequality constraints in Banach spaces and got some criteria and characterizations for these Levitin–Polyak well-posedness. They also investigated the relationships among the Levitin–Polyak well-posedness of the problem and the existence, uniqueness of its solution. Lemaire et al. (2002) introduced the well-posedness by perturbations for variational inequalities and obtained some metric characterizations for this well-posedness. In 2010, Fang et al. (2010) considered the well-posedness by perturbations for mixed variational inequality problems in Banach spaces and derived some metric characterizations of the well-posedness by perturbations. They established the equivalence between the well-posedness of mixed variational inequalities and that of a corresponding inclusion problem and a fixed point problem and obtained the relationship among the well-posedness by perturbations and the existence and uniqueness of its solution.

Very recently, Huang et al. (2007) and Li et al. (2006) studied the generalized vector equilibrium problems. By virtue of a nonlinear scalarization function, the gap functions for generalized vector equilibrium problems were established. Further, existence theorems for generalized vector equilibrium problems are derived by using the gap function. Li and Li (2009) also introduced two type of Levitin–Polyak well-posedness for equilibrium problems with abstract set constraints. Motivated by Li and Li (2009), Peng et al. (2009) introduced four type of Levitin–Polyak well-posedness for vector equilibrium problems with abstract set and functional constraints. Hu et al. (2010b) investigated the well-posedness and generalized well-posedness for a system of equilibrium problems, obtained some metric characterizations for these well-posedness. They also proved that the well-posedness of system of equilibrium problems is equivalent to the existence and uniqueness of its solution. Peng and Wu (2010) also explored the generalized Tykhonov well-posedness for system of vector quasi-equilibrium problems and gave some metric characterizations for these well-posedness in locally convex Hausdorff topological vector spaces.

However, to the best of our knowledge, there are little results concerning the existence of solutions and Levitin–Polyak well-posedness by perturbations for general systems of set-valued vector quasi-equilibrium problems [for short, (SSVQEP)].

Inspired and motivated by the researches going on in this direction, the aim of this paper is to introduce and investigate Levitin–Polyak well-posedness by perturbations for a new class of (SSVQEP) in Hausdorff topological vector spaces.

Firstly, existence theorems of solutions for the system of set-valued vector quasi-equilibrium problem with respect to a parameter [for short, (PSSVQEP)] and its dual problem (DPSSVQEP) are established under some suitable conditions.

Secondly, we introduce the notions of type I (resp., type II, generalized type I and generalized type II) Levitin–Polyak well-posedness by perturbations for (SSVQEP) in topological vector spaces. Some metric characterizations of the type I (resp., type II, generalized type I and generalized type II) Levitin–Polyak well-posedness by perturbations are derived under some suitable conditions. We also explore the relationships among the type I (resp., type II, generalized type I and generalized type II) Levitin–Polyak well-posedness by perturbations, the existence and uniqueness of solution to (SSVQEP). By virtue of the nonlinear scalarization function introduced by Chen et al. (2005a), a parametric gap function g for (PSSVQEP) is introduced, which is distinct from that of Peng et al. (2012), and then the continuity of the parametric gap function g is derived under quite mild assumptions.

Finally, we establish the equivalence between the type I (resp., type II, generalized type I and generalized type II) Levitin–Polyak well-posedness by perturbations of (SSVQEP) and that of a corresponding minimization problem with functional constraints under quite mild assumptions. The results presented in this paper are new, unifying and improving some known results in the literature.

2 Preliminaries

Throughout this paper, without other specifications, let I be an index set, R be the set of real numbers. Let $\bigwedge$ (the space of parameters) be a metric space, Z_i be topological vector space, X_i and Y_i be two locally convex Hausdorff topological vector spaces for $i \in I$, H_i and K_i be nonempty convex subsets of X_i and Y_i, respectively. Let $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$, $H = \prod_{i\in I} H_i$, $K = \prod_{i\in I} K_i$ and $X^{-i} = \prod_{j\in I, j\neq i} X_j$. Denote the element of X^{-i} by x^{-i} and so $x \in X$ denote by $x = (x^{-i}, x_i) \in X^{-i} \times X_i$. We always denote 2^X by the family of all nonempty subsets of X. Let $\Gamma_i : H \to 2^{H_i}$, $T_i : H \to 2^{K_i}$, $\Psi_i : H \times H_i \to 2^{Z_i}$, $F_i : H \times K \times H_i \to 2^{Z_i}$ and $C_i : H \to 2^{Z_i}$ be set-valued mappings such that, for each $x \in H$, $C_i(x)$ is a proper closed convex and pointed cone in Z_i with $\text{int}C_i(x) \neq \emptyset$ for each $i \in I$.

We consider the following system of set-valued vector quasi-equilibrium problem [for short, (SSVQEP)]:

Find $x^* = (x_i^*)_{i\in I} \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \not\subseteq -\text{int}C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*), \tag{2.1}$$

and the dual system of set-valued vector quasi-equilibrium problem [for short, (DSSVQEP)]:

Find $x^* = (x_i^*)_{i\in I} \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \subseteq -C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*). \tag{2.2}$$

If the mappings $\Gamma_i : H \to 2^{H_i}$ and $T_i : H \to 2^{K_i}$ are perturbed by a parameter $p \in \bigwedge$, that is, $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ and $T_i : H \times \bigwedge \to 2^{K_i}$ such that, for some $p^* \in \bigwedge$, $\Gamma_i(x, p^*) = \Gamma_i(x)$ and $T_i(x, p^*) = T_i(x)$ for all $x \in H$, then, for any given $p \in \bigwedge$, we define a parametric system of set-valued vector quasi-equilibrium problem [for short, (PSSVQEP)]:

Find $x^* = (x_i^*)_{i\in I} \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$ and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \not\subseteq -\text{int}C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p),$$

and the dual parametric system of set-valued vector quasi-equilibrium problem [for short, (DPSSVQEP)]:

Find $x^* = (x_i^*)_{i\in I} \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$ and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \subseteq -C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p).$$

We denote the solution sets of (SSVQEP), (DSSVQEP), (PSSVQEP) and (DPSSVQEP) by S, S^d, $S(p)$ and $S(p)^d$, respectively.

Special cases are as follows:

(1) If, for each $i \in I$, $\Psi_i \equiv 0$, then (SSVQEP) and (DSSVQEP) are reduced to the following:

Find $x^* \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$F_i(x^*, y^*, x_i) \nsubseteq -\text{int}C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*), \tag{2.3}$$

and its dual problem:

Find $x^* \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$F_i(x^*, y^*, x_i) \subseteq -C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*), \tag{2.4}$$

which have been studied by Homidan et al. (2007) and references therein.

(2) If, for each $i \in I$, $F_i \equiv 0$, then (SSVQEP) and (DSSVQEP) are reduced to the following:

Find $x^* \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$\Psi_i(x^*, x_i) \nsubseteq -\text{int}C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*), \tag{2.5}$$

and its dual problem:

Find $x^* \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$\Psi_i(x^*, x_i) \subseteq -C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*). \tag{2.6}$$

These problems have been studied by Ansari et al. (2002), Fang et al. (2006) and references therein.

(3) If, for each $i \in I$, $\Psi_i \equiv 0$, the mapping F_i reduces to a single-valued mapping, then (SSVQEP) and (DSSVQEP) are reduced to the following:

Find $x^* \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$F_i(x^*, y^*, x_i) \notin -\text{int}C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*), \tag{2.7}$$

and its dual problem:

Find $x^* \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*)$ and there exists $y_i^* \in T_i(x^*)$ satisfying

$$F_i(x^*, y^*, x_i) \in -C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*), \tag{2.8}$$

which have been studied by Ansari et al. (2000) and references therein.

(4) If the index set I is a singled set, $F_I \equiv 0$, then (SSVQEP) and (DSSVQEP) are reduced to the following:

Find $x^* \in H$ such that $x^* \in \Gamma_I(x^*)$ and

$$\Psi_I(x^*, x) \nsubseteq -\text{int}C_I(x^*), \quad \forall x \in \Gamma_I(x^*), \tag{2.9}$$

and its dual problem:
Find $x^* \in H$ such that $x^* \in \Gamma_I(x^*)$ and

$$\Psi_I(x^*, x) \subseteq -C_I(x^*), \quad \forall x \in \Gamma_I(x^*), \tag{2.10}$$

which have been studied by Fu (2005) and references therein.

(5) If the index set I is a singled set, $\Psi_I \equiv 0$, the mapping F_I reduces to a single-valued mapping, then (SSVQEP) is reduced to the following:

Find $x^* \in H$ such that $x^* \in \Gamma_I(x^*)$ and there exists $y^* \in T_I(x^*)$ satisfying

$$F_I(x^*, y^*, x) \notin -\text{int}C_I(x^*), \quad \forall x \in \Gamma_I(x^*), \tag{2.11}$$

which has been studied by Peng et al. (2012) and references therein.

In brief, for appropriate choice of I and the mappings $\Gamma_i, T_i, F_i, \Psi_i$, one can get a wide class of variational models such as variational inequalities (inclusions), fixed point problems, optimization problems and so on. These suffice that (SSVQEP) and (DSSVQEP) are more general and include some classes of variational problems and related optimization problems as special cases.

We first recall some definitions and lemmas which are needed in our main results.

Definition 2.1 Let E_1, E_2 be topological vector spaces, $\Delta : E_1 \times E_1 \to 2^{E_2}$ and $C : E_1 \to 2^{E_2}$ be set-valued mappings such that, for each $x \in E_1$, $C(x)$ is a proper closed convex and pointed cone in E_2 with $\text{int}C(x) \neq \emptyset$. Δ is said to be *generalized C-convex* with respect to the second argument if, for any $x, \hat{x}, \tilde{x} \in E_1$ and $t \in [0, 1]$,

$$\Delta(x, t\hat{x} + (1-t)\tilde{x}) \subseteq t\Delta(x, \hat{x}) + (1-t)\Delta(x, \tilde{x}) - C(x).$$

Definition 2.2 (Chen et al. 2005b,a) Let E_1, E_2 be locally convex Hausdorff topological vector spaces, $C : E_1 \to 2^{E_2}$ be a set-valued mapping such that, for each $x \in E_1$, $C(x)$ is a proper closed convex and pointed cone in E_2 with $\text{int}C(x) \neq \emptyset$. The *nonlinear scalarization function* $\xi_e : E_1 \times E_2 \to R$ is defined by

$$\xi_e(x, y) = \inf\{z \in R : y \in ze(x) - C(x)\}, \quad \forall (x, y) \in E_1 \times E_2,$$

where $e : E_1 \to E_2$ is a vector-valued mapping and $e(x) \in \text{int}C(x)$ for all $x \in E_1$.

Lemma 2.1 (Chen et al. 2005b,a) *Let E_1, E_2 be locally convex Hausdorff topological spaces, $C : E_1 \to 2^{E_2}$ be a set-valued mapping such that, for each $x \in E_1$, $C(x)$ is a proper closed convex and pointed cone in E_2 with $\text{int}C(x) \neq \emptyset$ and $e : E_1 \to E_2$ be* *vector-valued such that $e(x) \in intC(x)$ for all $x \in E_1$. For any $x \in E_1$, $y \in E_2$ and $r \in R$, the following results hold:*

(1) $\xi_e(x, y) < r \Leftrightarrow y \in re(x) - intC(x)$;
(2) $\xi_e(x, y) \leq r \Leftrightarrow y \in re(x) - C(x)$;
(3) $\xi_e(x, y) \geq r \Leftrightarrow y \notin re(x) - intC(x)$;
(4) $\xi_e(x, y) > r \Leftrightarrow y \notin re(x) - C(x)$;
(5) $\xi_e(x, y) = r \Leftrightarrow y \in re(x) - \partial C(x)$, *particularly,* $\xi_e(x, re(x)) = r$ *and* $\xi_e(x, 0) = 0$, *where* $\partial C(x)$ *is the boundary of* $C(x)$.

Lemma 2.2 (Chen et al. 2000, 2005a) *Let E_1, E_2 be two locally convex Hausdorff topological vector spaces, and let $C : E_1 \to 2^{E_2}$ be a set-valued mapping such that, for each $x \in E_1$, $C(x)$ is a proper closed convex and pointed cone in E_2 with $intC(x) \neq \emptyset$, Let $e : E_1 \to E_2$ be a continuous selection from the set-valued map $intC(\cdot)$. Define a set-valued mapping $\Theta : E_1 \to 2^{E_2}$ by $\Theta(x) = E_2 \setminus intC(x)$ for any $x \in E_1$. Then we have the following:*

(1) *If the mappings $C(\cdot)$ and $\Theta(\cdot)$ are B-u.s.c on E_1, then $\xi_e(\cdot, \cdot)$ is continuous on $E_1 \times E_2$;*
(2) *The mapping $\xi_e(x, \cdot) : E_2 \to R$ is convex;*
(3) *If $\Theta(\cdot)$ is B-u.s.c on E_1, then $\xi_e(\cdot, \cdot)$ is upper semicontinuous on $E_1 \times E_2$;*
(4) *If $C(\cdot)$ is B-u.s.c on E_1, then $\xi_e(\cdot, \cdot)$ is lower semicontinuous on $E_1 \times E_2$.*

Definition 2.3 (Kuratowski 1968) Let A and B be nonempty subsets of X. The *Hausdorff metric* $\mathscr{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathscr{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ is the *excess* of set A to set B and $d(a, B) = \inf_{b \in B} \|a - b\|$.

Definition 2.4 (Kuratowski 1968) Let A be a nonempty subset of X. The *Kuratowski measure of noncompactness* $\mathscr{M}$ of the set A is defined by

$$\mathscr{M}(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i, \text{diam} A_i < \epsilon,\ i = 1, 2, \ldots, n\},$$

where diam stands for the diameter of a set.

Remark 2.1 (Fang et al. 2010) If $\mathscr{L}_1, \mathscr{L}_2$ are nonempty closed subset of X, $\mathscr{L}_1$ is compact and $\mathscr{L}_1 \subseteq \mathscr{L}_2$, then the following hold:

(1) $\mathscr{M}(\mathscr{L}_1) = 0$;
(2) $\mathscr{M}(\mathscr{L}_2) \leq 2\mathscr{H}(\mathscr{L}_2, \mathscr{L}_1) = 2e(\mathscr{L}_2, \mathscr{L}_1)$.

Definition 2.5 (Aubin and Ekeland 1984; Berge 1963) Let $\bigvee$ be a Hausdorff topological vector space and E be a locally convex Hausdorff topological vector space. A set-valued mapping $\psi : \bigvee \to 2^E$ is said to be:

(1) *upper semicontinuous* (for short, u.s.c) at $\upsilon_0 \in \bigvee$ if, for each open set V with $\psi(\upsilon_0) \subset V$, there exists $\delta > 0$ such that

$$\psi(\upsilon) \subset V, \quad \forall \upsilon \in B(\upsilon_0, \delta);$$

(2) *lower semicontinuous* (for short, l.s.c) at $\upsilon_0 \in \bigvee$ if, for each open set V with $\psi(\upsilon_0) \bigcap V \neq \emptyset$, there exists $\delta > 0$ such that

$$\psi(\upsilon) \cap V \neq \emptyset, \quad \forall \upsilon \in B(\upsilon_0, \delta);$$

(3) *closed* if the graph of ψ is closed, i.e., the set $Gr(\psi) = \{(\zeta, \upsilon) \in \bigvee \times E : \zeta \in \psi(\upsilon)\}$ is closed in $\bigvee \times E$;
(4) *lower semicontinuous* (resp., *upper semicontinuous*) on $\bigvee$ if it is l.s.c (resp. u.s.c) at each $\upsilon \in \bigvee$;
(5) *continuous* on $\bigvee$ if it is both l.s.c and u.s.c on $\bigvee$.

Remark 2.2 (Aubin and Ekeland 1984; Berge 1963)

(1) ψ is l.s.c at $\upsilon_0 \in \bigvee$ if and only if, for any net $\{\upsilon_\alpha\} \subseteq \bigvee$ with $\upsilon_\alpha \to \upsilon_0$ and $\zeta_0 \in \psi(\upsilon_0)$, there exists a net $\{\zeta_\alpha\} \subseteq E$ with $\zeta_\alpha \in \psi(\upsilon_\alpha)$ for all α such that $\zeta_\alpha \to \zeta_0$.
(2) If ψ is compact-valued, then ψ is u.s.c at $\upsilon_0 \in \bigvee$ if and only if, for any net $\{\upsilon_\alpha\} \subseteq \bigvee$ with $\upsilon_\alpha \to \upsilon_0$ and $\{\zeta_\alpha\} \subseteq E$ with $\zeta_\alpha \in \psi(\upsilon_\alpha)$ for all α, there exists $\zeta_0 \in \psi(\upsilon_0)$ and a subnet $\{\zeta_\beta\}$ of $\{\zeta_\alpha\}$ such that $\zeta_\beta \to \zeta_0$.
(3) If ψ is u.s.c and closed-valued, then ψ is closed. Conversely, if ψ is closed and E is compact, then ψ is u.s.c.

Definition 2.6 (Ding et al. 1992; Li and He 2005) Let $\phi : X \to Z$ be vector-valued mapping and $\Phi : X \to 2^Z$ be set-valued mapping.

(1) ϕ is called a *selection* of Φ on X if

$$\phi(x) \in \Phi(x), \quad \forall x \in X.$$

(2) ϕ is called a *continuous selection* of Φ on X if ϕ is a selection of Φ and continuous on X.

Remark 2.3 If Φ is a continuous set-valued mapping on X, then there exists a selection ϕ of Φ which is continuous on X. For more information on selection and continuous selection of set-valued mappings see, for instance, Ding et al. (1992), Chen (1988), Gutev (1998) and the references therein.

Lemma 2.3 (Ansari et al. 2000) *Let H_i be nonempty convex subset of Hausdorff topological vector space X_i for each $i \in I$ and $M_i : H \to 2^{H_i}$ be a convex-valued mapping for each $i \in I$, where $H = \prod_{i \in I} H_i$. Assume that the following conditions hold:*

(i) *for each $x \in H$, $x_i \notin M_i(x)$ for each $i \in I$;*
(ii) *for each $x_i \in H_i$, $M_i^{-1}(x_i)$ is an open subset of H for each $i \in I$;*

(iii) *there exist a nonempty compact subset* Ω *of* H *for each* $i \in I$ *and a nonempty compact convex subset* $\mathscr{Z}_i$ *of* H_i *such that, for any* $x \in H \setminus \Omega$, *there exists* $j \in I$ *satisfying*

$$M_j(x) \bigcap \mathscr{Z}_j \neq \emptyset.$$

Then there exists $x^* \in H$ such that $M_i(x^*) = \emptyset$ for each $i \in I$.

3 Existence theorems for (PSSVQEP)

In this section, we mainly study the existence of solution to the parametric systems of set-valued vector quasi-equilibrium problems [for short, (PSSVQEP)] and its dual (DPSSVQEP) under some suitable conditions.

Theorem 3.1 *Let* Λ *be a metric space and, for each* $i \in I$, *let* Z_i *be topological vector space,* H_i *and* K_i *be nonempty closed convex subsets of locally convex Hausdorff topological vector spaces* X_i *and* Y_i, *respectively. Let* $F_i : H \times K \times H_i \to 2^{Z_i}$, $C_i : H \to 2^{Z_i}$ *be two set-valued mappings such that, for each* $x \in H$, $C_i(x)$ *is a proper closed convex and pointed cone in* Z_i *with* $intC_i(x) \neq \emptyset$, $\Gamma_i : H \times \Lambda \to 2^{H_i}$, $T_i : H \times \Lambda \to 2^{K_i}$ *be two nonempty convex-valued mappings and* $\Psi_i : H \times H_i \to 2^{Z_i}$ *be compact-valued and upper semicontinuous with respect to the first argument and generalized* C_i*-convex with respect to the second argument, where* $H = \prod_{i \in I} H_i$ *and* $K = \prod_{i \in I} K_i$. *Assume that the following conditions are satisfied:*

(i) *for each* $i \in I$, $x_i \in H_i$, $y_i \in K_i$, $\Gamma_i^{-1}(x_i)$ *and* $T_i^{-1}(y_i)$ *are open sets of* $H \times \Lambda$;
(ii) *for each* $i \in I$, $W_i(\cdot) = Z_i \setminus -intC_i(\cdot)$ *is closed on* H;
(iii) *for each* $i \in I$, $x_i' \in H_i$, *the mapping* $(x, y) \mapsto F_i(x, y, x_i')$ *is compact-valued and u.s.c. and, for each* $x \in H$ *and* $y_i \in T_i(x)$, *the mapping* $x_i' \mapsto F_i(x, y, x_i')$ *is generalized* C_i*-convex;*
(iv) *there exist nonempty compact sets* $\Omega \subseteq H$, $\Xi \subseteq K$ *and nonempty compact convex sets* $U_i \subseteq H_i$, $L_i \subseteq K_i$ *for each* $i \in I$ *such that, for any* $(x, y) \in H \times K \setminus (\Omega \times \Xi)$, *there exists* $i' \in I$ *with* $x_{i'} \in U_{i'} \bigcap \Gamma_{i'}(x, p)$ *and* $y_{i'} \in L_{i'} \bigcap T_{i'}(x, p)$ *for any* $p \in \Lambda$ *satisfying*

$$F_{i'}(x, y, x_{i'}) + \Psi_{i'}(x, x_{i'}) \subseteq -intC_{i'}(x).$$

Then, for each $p \in \Lambda$, *the solution set* $S(p)$ *of (PSSVQEP) is nonempty.*

Proof Let $p \in \Lambda$. For each $i \in I$, we define two set-valued mappings $\Upsilon_i : H \times K \to 2^{H_i}$ and $\Phi_i : H \times K \to 2^{H_i \times K_i}$ by, for any $(x, y) \in H \times K$,

$$\Upsilon_i(x, y) = \{\omega_i \in H_i : F_i(x, y, \omega_i) + \Psi_i(x, \omega_i) \subseteq -\text{int}C_i(x)\},$$

and

$$\Phi_i(x, y) = \begin{cases} (\Gamma_i(x, p) \bigcap \Upsilon_i(x, y)) \times T_i(x, p), & (x, y) \in \{(x, y) : x_i \in \Gamma_i(x, p), y_i \in T_i(x, p)\}, \\ \Gamma_i(x, p) \times T_i(x, p), & (x, y) \notin \{(x, y) : x_i \in \Gamma_i(x, p), y_i \in T_i(x, p)\}, \end{cases}$$

respectively. It follows that the properties of Φ_i are related to that of Υ_i.

Now, we show that, for each $i \in I$, $\omega_i \in H_i$, Υ_i is convex-valued and $\Upsilon_i^{-1}(\omega_i)$ is an open subset of $H \times K$. For this, we first prove that Υ_i is convex-valued. For each $(x, y) \in H \times K$, taking $\tilde{\omega}_i, \hat{\omega}_i \in \Upsilon_i(x, y)$ arbitrarily, one has

$$F_i(x, y, \tilde{\omega}_i) + \Psi_i(x, \tilde{\omega}_i) \subseteq -\mathrm{int}C_i(x), \tag{3.1}$$
$$F_i(x, y, \hat{\omega}_i) + \Psi_i(x, \hat{\omega}_i) \subseteq -\mathrm{int}C_i(x), \tag{3.2}$$

and so $t\tilde{\omega}_i + (1-t)\hat{\omega}_i \in H_i$ for all $t \in [0, 1]$, since H_i is convex. By the generalized C_i-convexity of Ψ_i with respect to the second argument, we have

$$\Psi_i(x, t\tilde{\omega}_i + (1-t)\hat{\omega}_i) \subseteq t\Psi_i(x, \tilde{\omega}_i) + (1-t)\Psi_i(x, \hat{\omega}_i) - C_i(x). \tag{3.3}$$

It follows from the condition (iii) that

$$F_i(x, y, t\tilde{\omega}_i + (1-t)\hat{\omega}_i) \subseteq tF_i(x, y, \tilde{\omega}_i) + (1-t)F_i(x, y, \hat{\omega}_i) - C_i(x). \tag{3.4}$$

Thus, from (3.1)–(3.4), it follows that

$$\begin{aligned}
&\Psi_i(x, t\tilde{\omega}_i + (1-t)\hat{\omega}_i) + F_i(x, y, t\tilde{\omega}_i + (1-t)\hat{\omega}_i) \\
&\quad \subseteq t(F_i(x, y, \tilde{\omega}_i) + \Psi_i(x, \tilde{\omega}_i)) + (1-t)(F_i(x, y, \hat{\omega}_i) + \Psi_i(x, \hat{\omega}_i)) - C_i(x) \\
&\quad \subseteq -t\mathrm{int}C_i(x) - (1-t)\mathrm{int}C_i(x) - C_i(x) \\
&\quad \subseteq -\mathrm{int}C_i(x).
\end{aligned}$$

Therefore, $t\tilde{\omega}_i + (1-t)\hat{\omega}_i \in \Upsilon_i(x, y)$ for all $t \in [0, 1]$, that is, $\Upsilon_i(x, y)$ is a convex subset of H_i.

Next, we prove that, for each $i \in I$ and $\omega_i \in H_i$, $\Upsilon_i^{-1}(\omega_i)$ is an open subset of $H \times K$. It suffices that the complementary set of $\Upsilon_i^{-1}(\omega_i)$, denoted by

$$\left[\Upsilon_i^{-1}(\omega_i)\right]^c = \{(x, y) \in H \times K : F_i(x, y, \omega_i) + \Psi_i(x, \omega_i) \nsubseteq -\mathrm{int}C_i(x)\},$$

is an closed subset of $H \times K$. Take a sequence $\{(x_n, y_n)\} \subseteq [\Upsilon_i^{-1}(\omega_i)]^c$ such that $(x_n, y_n) \to (x_0, y_0)$. Since $H \times K$ is closed, we obtain $(x_0, y_0) \in H \times K$. Now, we assert that $(x_0, y_0) \in [\Upsilon_i^{-1}(\omega_i)]^c$. If not, $(x_0, y_0) \notin [\Upsilon_i^{-1}(\omega_i)]^c$ and so we have

$$F_i(x_0, y_0, \omega_i) + \Psi_i(x_0, \omega_i) \subseteq -\mathrm{int}C_i(x_0). \tag{3.5}$$

In view of $\{(x_n, y_n)\} \subseteq [\Upsilon_i^{-1}(\omega_i)]^c$, one can conclude

$$F_i(x_n, y_n, \omega_i) + \Psi_i(x_n, \omega_i) \nsubseteq -\mathrm{int}C_i(x_n). \tag{3.6}$$

Thus there exist $\varsigma_i^n \in F_i(x_n, y_n, \omega_i)$ and $\tau_i^n \in \Psi_i(x_n, \omega_i)$ such that

$$\varsigma_i^n + \tau_i^n \notin -\mathrm{int}C_i(x_n),$$

that is,

$$\varsigma_i^n + \tau_i^n \in W_i(x_n). \tag{3.7}$$

Since $\Psi_i : H \times H_i \to 2^{Z_i}$ is compact-valued and upper semicontinuous with respect to the first argument, it follows from the condition (iii) that there exist $\varsigma_i^0 \in F_i(x_0, y_0, \omega_i)$ and $\tau_i^0 \in \Psi_i(x_0, \omega_i)$ such that $\varsigma_i^n \to \varsigma_i^0$ and $\tau_i^n \to \tau_i^0$, respectively. By the condition (ii), we know that $(x_0, \varsigma_i^0 + \tau_i^0) \in Gr(W)$, which implies that

$$\varsigma_i^0 + \tau_i^0 \notin -\text{int}C_i(x_0). \tag{3.8}$$

This contradicts (3.5). Consequently, $[\Upsilon_i^{-1}(\omega_i)]^c$ is a closed set and so $\Upsilon_i^{-1}(\omega_i)$ is an open set in $H \times K$.

Finally, we verify that Φ_i satisfies the conditions of Lemma 2.3 for each $i \in I$. From the definition of Φ_i, it is easy to see that, for each $i \in I$, $(x, y) \in H \times K$, $\Phi_i(x, y)$ is a convex subset of $H_i \times K_i$ and so $(x_i, y_i) \notin \Phi_i(x, y)$. By the condition (i), we know that, for each $i \in I$, $(\bar{x}_i, \bar{y}_i) \in H_i \times K_i$, $\Phi_i^{-1}(\bar{x}_i, \bar{y}_i)$ is an open subset of $H \times K$. By the condition (iv), there exist nonempty compact sets $\Omega \subseteq H$, $\Xi \subseteq K$ and nonempty compact convex sets $U_i \subseteq H_i$, $L_i \subseteq K_i$ for each $i \in I$ such that, for any $(x, y) \in H \times K \setminus (\Omega \times \Xi)$, there exists $i' \in I$ with $(x_{i'}, y_{i'}) \in (U_{i'} \times L_{i'}) \bigcap \Phi_{i'}(x, y)$, that is, $(U_{i'} \times L_{i'}) \bigcap \Phi_{i'}(x, y) \neq \emptyset$. Again, from Lemma 2.3, this yields that there exists $(x^*, y^*) \in H \times K$ such that

$$\Phi_i(x^*, y^*) = \emptyset, \quad \forall i \in I. \tag{3.9}$$

Since $\Gamma_i : H \times \Lambda \to 2^{H_i}$ and $T_i : H \times \Lambda \to 2^{K_i}$ are two nonempty convex-valued mappings for each $i \in I$, we have

$$(x^*, y^*) \in \{(x, y) : x_i \in \Gamma_i(x, p), y_i \in T_i(x, p)\}, \quad \forall i \in I,$$

and

$$\left(\Gamma_i(x^*, p) \bigcap \Upsilon_i(x^*, y^*)\right) \times T_i(x^*, p) = \emptyset, \quad \forall i \in I.$$

Moreover, one has

$$\Upsilon_i(x^*, y^*) = \emptyset, \quad \forall i \in I,$$

Therefore, for each $p \in \Lambda$, there exists $x^* \in H$ such that, for each $i \in I$, $x_i^* \in \Gamma_i(x^*, p)$ and there exists $y_i^* \in T_i(x^*, p)$ satisfying

$$F_i(x^*, y^*, x_i) + \Psi_i(x^*, x_i) \nsubseteq -\text{int}C_i(x^*), \quad \forall x_i \in \Gamma_i(x^*, p),$$

that is, for each $p \in \Lambda$, the solutions set $S(p)$ of (PSSVQEP) is nonempty. This completes the proof. □

Similarly, we can show that the solution set for (DPSSVQEP) is nonempty.

Theorem 3.2 *Let Λ be a metric space and, for each $i \in I$, Z_i be topological vector space, H_i and K_i be nonempty convex subsets of locally convex Hausdorff topological vector spaces X_i and Y_i, respectively. Let $F_i : H \times K \times H_i \to 2^{Z_i}, C_i : H \to 2^{Z_i}$ be two set-valued mappings such that C_i is lower semicontinuous and, for each $x \in H, C_i(x)$ is a proper closed convex and pointed cone in Z_i with $intC_i(x) \neq \emptyset, \Gamma_i : H \times \Lambda \to 2^{H_i}, T_i : H \times \Lambda \to 2^{K_i}$ be two closed convex-valued mappings and $\Psi : H \times H_i \to 2^{Z_i}$ be lower semicontinuous with respect to the first argument and generalized $-C_i$-convex with respect to the second argument. Assume that the following conditions are satisfied:*

(i) *for each $i \in I, x_i \in H_i, y_i \in K_i, \Gamma_i^{-1}(x_i)$ and $T_i^{-1}(y_i)$ are open sets of $H \times \Lambda$;*
(ii) *for each $i \in I$, $x_i' \in H_i$, the mapping $(x, y) \mapsto F_i(x, y, x_i')$ is lower semicontinuous and, for each $x \in H$ and $y_i \in T_i(x)$, the mapping $x_i' \mapsto F_i(x, y, x_i')$ is generalized $-C_i$-convex;*
(iii) *there exist nonempty compact sets $\Omega \subseteq H$, $\Xi \subseteq K$ and nonempty compact convex sets $U_i \subseteq H_i, L_i \subseteq K_i$ for each $i \in I$ such that, for any $(x, y) \in H \times K \setminus (\Omega \times \Xi)$, there exists $i' \in I$ with $x_{i'} \in U_{i'} \bigcap \Gamma_{i'}(x, p)$ and $y_{i'} \in L_{i'} \bigcap T_{i'}(x, p)$ for each $p \in \Lambda$ satisfying*

$$F_{i'}(x, y, x_{i'}) + \Psi_{i'}(x, x_{i'}) \nsubseteq -C_{i'}(x).$$

Then, for each $p \in \Lambda$, the solutions set $S(p)^d$ of (DPSSVQEP) is nonempty.

Proof Let $p \in \Lambda$. For each $i \in I$, we define the mapping $\Upsilon_i : H \times K \to 2^{H_i}$ by

$$\Upsilon_i(x, y) = \{\omega_i \in H_i : F_i(x, y, \omega_i) + \Psi_i(x, \omega_i) \nsubseteq -C_i(x)\}, \quad \forall (x, y) \in H \times K.$$

The rest proof is similar to that of Theorem 3.1 and so we omit it here. This completes the proof. □

Remark 3.1 If $p = p^*$, then it follows from Theorems 3.1 and 3.2 that the solution sets of (SSVQEP) and (DSSVQEP) are nonempty.

4 Levitin–Polyak well-posedness for (SSVQEP)

In this section, we introduce the notions of type I (resp., type II, generalized type I and generalized type II) Levitin–Polyak well-posedness by perturbations for (SSVQEP) and discuss some metric characterizations of these Levitin–Polyak well-posedness and the relationships between these Levitin–Polyak well-posedness and the existence and uniqueness of a solution to (SSVQEP).

Definition 4.1 Let $\bigwedge$ be a metric space, and $\{p^n\} \subset \bigwedge$ such that $p^n \to p^*$.

(1) A sequence $\{x^n\} \subset H$ is said to be the *type I Levitin–Polyak (for short, LP) approximating solution sequence* corresponding to $\{p^n\}$ for (SSVQEP) if, for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence of nonnegative real numbers $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i\left(x^n, y^n, \omega_i\right) + \Psi_i\left(x^n, \omega_i\right) + \epsilon_n e_i\left(x^n\right) \nsubseteq -\mathrm{int}C_i(x^n),$$
$$\forall \omega_i \in \Gamma_i\left(x^n, p^n\right), n \in N.$$

(2) A sequence $\{x^n\} \subset H$ is said to be the *type II LP approximating solution sequence* corresponding to $\{p^n\}$ for (SSVQEP) if there exist a sequence of nonnegative real numbers $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ for each $i \in I$ such that

$$d_i\left(x_i^n, \Gamma_i\left(x^n, p^n\right)\right) \leq \epsilon_n$$

and

$$F_i\left(x^n, y^n, \omega_i\right) + \Psi_i\left(x^n, \omega_i\right) + \epsilon_n e_i\left(x^n\right) \nsubseteq -\mathrm{int}C_i\left(x^n\right),$$
$$\forall \omega_i \in \Gamma_i\left(x^n, p^n\right), n \in N.$$

Remark 4.1 It is easy to see that any type I LP approximating solution sequence corresponding to $\{p^n\}$ for (SSVQEP) is the type II LP approximating solution sequence corresponding to $\{p^n\}$ for (SSVQEP).

Definition 4.2 (1) (SSVQEP) is said to be the *type I* (resp., *type II*) *LP well-posed by perturbations* if it has a unique solution and, for any $\{p^n\} \subset \bigwedge$ with $p^n \to p^*$, each type I (resp., type II) LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP) converges strongly to the unique solution.

(2) (SSVQEP) is said to be the *generalized type I* (resp., *type II*) *LP well-posed by perturbations* if the solution set S of (SSVQEP) is nonempty and, for any $\{p^n\} \subset \bigwedge$ with $p^n \to p^*$, each type I (resp., type II) LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP) has a subsequence which converges strongly to some point of S.

Remark 4.2 (1) The type I (resp., type II, generalized type I and generalized type II) LP well-posedness by perturbations for (SSVQEP) implies that the solution set S of (SSVQEP) is nonempty and compact.

(2) Each type I LP well-posedness by perturbations for (SSVQEP) is the type II (resp., generalized type I and generalized type II) LP well-posedness by perturbations for (SSVQEP). Moreover, any generalized type I LP well-posedness by perturbations for (SSVQEP) is the generalized type II LP well-posedness by perturbations for (SSVQEP).

Remark 4.3 (1) If, for each $i \in I$, the mappings $T_i \equiv 0$, $F_i \equiv 0$, $\Gamma_i(x, p) = \Gamma_i(x)$ for all $(x, p) \in X \times \bigwedge$ and Ψ_i is a single-valued mapping, then the generalized type II LP well-posedness by perturbations for (SSVQEP) is reduced to the generalized Tykhonov well-posedness defined by Peng and Wu (2010).

(2) If I is a single set, the mappings $T_i(x, p) \equiv T(x), \Gamma_i(x, p) \equiv \Gamma(x)$ for all $(x, p) \in X \times \bigwedge, \Psi_I \equiv 0$ and $F_I : H \times K \times H \to Z_I$ is a vector-valued mapping, then the (generalized) type II LP well-posedness by perturbations for (SSVQEP) is reduced to the type I LP well-posedness for generalized vector equilibrium problem defined by Peng et al. (2012).

In order to investigate the metric characterizations of well-posedness of (SSVQEP), for any $\epsilon > 0$, we introduce the following approximating solution sets for (SSVQEP):

$$\Omega_1(\epsilon) = \bigcup_{p \in B(p^*,\epsilon)} \{x \in H : \forall i \in I,\ x_i \in \Gamma_i(x, p),\ \exists y_i \in T_i(x, p) \text{ s.t.}$$
$$F_i(x, y, \omega_i) + \Psi_i(x, \omega_i) + \epsilon e_i(x) \not\subseteq -\text{int}C_i(x),\ \forall \omega_i \in \Gamma_i(x, p)\}$$

and

$$\Omega_2(\epsilon) = \bigcup_{p \in B(p^*,\epsilon)} \{x \in H : \forall i \in I,\ d_i(x_i, \Gamma_i(x, p)) \leq \epsilon,\ \exists y_i \in T_i(x, p) \text{ s.t.}$$
$$F_i(x, y, \omega_i) + \Psi_i(x, \omega_i) + \epsilon e_i(x) \not\subseteq -\text{int}C_i(x),\ \forall \omega_i \in \Gamma_i(x, p)\},$$

where $B(p^*, \epsilon)$ means the closed ball centered at p^* with radius ϵ.

Clearly, $\Omega_1(\epsilon) \subseteq \Omega_2(\epsilon)$ for any $\epsilon > 0$.

Next, we present the closedness of Ω_1, Ω_2 and the relationship between Ω_j and the solution set S of (SSVQEP) for $j = 1, 2$.

Lemma 4.1 *Let $\bigwedge$ be a finite dimensional space. For each $i \in I$, let $C_i : H \to 2^{Z_i}$ be a set-valued mappings such that, for each $x \in H$, $C_i(x)$ is a proper closed convex and pointed cone in Z_i with int$C_i(x) \neq \emptyset$, $e_i : H \to Z_i$ be a continuous vector valued mapping with $e_i(x) \in$ int$C_i(x)$ for any $x \in H$, the set-valued mappings $F_i : H \times K \times H_i \to 2^{Z_i}$, $\Psi_i : H \times H_i \to 2^{Z_i}$ be continuous, the set-valued mapping $T_i : H \times \bigwedge \to 2^{K_i}$ be upper semicontinuous and compact-valued and $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ be lower semicontinuous and closed. Assume further that the mapping $W_i(\cdot) = Z_i \setminus -intC_i(\cdot)$ is closed on H. Then the following statements hold:*

(i) *for each $j \in \{1, 2\}$, $\Omega_j(\epsilon)$ is closed for all $\epsilon \geq 0$;*
(ii) *$S = \bigcap_{\epsilon>0} \Omega_j(\epsilon)$ for $j = 1, 2$.*

Proof We first prove that $\Omega_2(\epsilon)$ is closed for all $\epsilon \geq 0$ and $S = \bigcap_{\epsilon>0} \Omega_2(\epsilon)$. We divide the proof into two steps.

Step 1. Let us show that, for each $\epsilon \geq 0$, $\Omega_2(\epsilon)$ is closed. Let $\{x^n\} \subset \Omega_2(\epsilon)$ and $x^n \to \hat{x}$. Then there exists $p^n \in B(p^*, \epsilon)$ such that, for each $i \in I$,

$$d_i\left(x_i^n, \Gamma_i\left(x^n, p^n\right)\right) \leq \epsilon \tag{4.1}$$

and there exists $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i\left(x^n, y^n, \omega_i\right) + \Psi_i\left(x^n, \omega_i\right) + \epsilon e_i\left(x^n\right) \not\subseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right),\ n \in N.$$

Since, for each $i \in I$, the set-valued mappings $F_i : H \times K \times H_i \to 2^{Z_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ are continuous, there exist continuous selections f_i and ψ_i of F_i and Ψ_i, respectively, such that

$$f_i\left(x^n, y^n, \omega_i\right) + \psi_i\left(x^n, \omega_i\right) + \epsilon e_i\left(x^n\right) \notin -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right). \tag{4.2}$$

Note that $W_i(\cdot) = Z_i \setminus -\mathrm{int}C_i(\cdot)$ is closed on H. From this and (4.2), it follows that

$$f_i\left(x^n, y^n, \omega_i\right) + \psi_i\left(x^n, \omega_i\right) + \epsilon e_i\left(x^n\right) \in W_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right). \quad (4.3)$$

Without loss of generality, suppose that $p^n \to \hat{p} \in B(p^*, \epsilon)$ since $B(p^*, \epsilon)$ is a closed ball and $\bigwedge$ is finite dimensional. Since $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is lower semicontinuous and closed, it follows from (4.1) that

$$d_i(\hat{x}_i, \Gamma_i(\hat{x}, \hat{p})) \leq \epsilon. \quad (4.4)$$

Again, from the upper semicontinuity and compactness of T_i for each $i \in I$, there exist a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ and $\hat{y}_i \in T_i(\hat{x}, \hat{p})$ such that $y_i^{n_k} \to \hat{y}_i$. Since $W_i(\cdot) = Z_i \setminus -\mathrm{int}C_i(\cdot)$ is closed on H and e_i, F_i, Ψ_i are continuous, it follows from (4.3) that

$$f_i(\hat{x}, \hat{y}, \omega_i) + \psi_i(\hat{x}, \omega_i) + \epsilon e_i(\hat{x}) \in W_i(\hat{x}), \quad \forall \omega_i \in \Gamma_i(\hat{x}, \hat{p}),$$

that is,

$$f_i(\hat{x}, \hat{y}, \omega_i) + \psi_i(\hat{x}, \omega_i) + \epsilon e_i(\hat{x}) \notin -\mathrm{int}C_i(\hat{x}), \quad \forall \omega_i \in \Gamma_i(\hat{x}, \hat{p}),$$

which implies that

$$F_i(\hat{x}, \hat{y}, \omega_i) + \Psi_i(\hat{x}, \omega_i) + \epsilon e_i(\hat{x}) \nsubseteq -\mathrm{int}C_i(\hat{x}), \quad \forall \omega_i \in \Gamma_i(\hat{x}, \hat{p})$$

and hence $\hat{x} \in \Omega_2(\epsilon)$, which implies that $\Omega_2(\epsilon)$ is closed for all $\epsilon \geq 0$.

Step 2. We prove that $S = \bigcap_{\epsilon > 0} \Omega_2(\epsilon)$. Clearly, $S \subseteq \bigcap_{\epsilon > 0} \Omega_2(\epsilon)$. Indeed, for each $\epsilon > 0, \epsilon e_i(x) \in \mathrm{int}C_i(x)$ for all $x \in H$. Suppose that there exists $\tilde{x} \in S$ such that $\tilde{x} \notin \Omega_2(\epsilon)$. Then, for each $i \in I$, $\tilde{x}_i \in \Gamma_i(\tilde{x}, p^*)$ and there exist $\tilde{y}_i \in T_i(\tilde{x}, p^*)$ and $\tilde{\omega}_i \in \Gamma_i(\tilde{x}, p^*)$ such that

$$F_i(\tilde{x}, \tilde{y}, \tilde{\omega}_i) + \Psi_i(\tilde{x}, \tilde{\omega}_i) \nsubseteq -\mathrm{int}C_i(\tilde{x}) \quad (4.5)$$

and

$$F_i(\tilde{x}, \tilde{y}, \tilde{\omega}_i) + \Psi_i(\tilde{x}, \tilde{\omega}_i) + \epsilon e_i(\tilde{x}) \subseteq -C_i(\tilde{x}).$$

Thus, from this, one has

$$F_i(\tilde{x}, \tilde{y}, \tilde{\omega}_i) + \Psi_i(\tilde{x}, \tilde{\omega}_i) \subseteq -C_i(\tilde{x}) - \epsilon e_i(\tilde{x}) \subseteq -\mathrm{int}C_i(\tilde{x}),$$

which contradicts (4.5).

Conversely, let $\bar{x} \in \bigcap_{\epsilon>0} \Omega_2(\epsilon)$. Then $\bar{x} \in \Omega_2(\epsilon)$ for all $\epsilon > 0$. Without loss of generality, let a sequence of real numbers $\{\epsilon_n\}$ with $\epsilon_n > 0$ and $\epsilon_n \to 0$. Thus $\bar{x} \in \Omega_2(\epsilon_n)$ and there exists $p^n \in B(p^*, \epsilon_n)$ such that, for each $i \in I$,

$$d_i\left(\bar{x}_i, \Gamma_i\left(\bar{x}, p^n\right)\right) \leq \epsilon_n \tag{4.6}$$

and there exists $\bar{y}_i^n \in T_i(\bar{x}, p^n)$ such that

$$F_i\left(\bar{x}, \bar{y}^n, \omega_i\right) + \Psi_i\left(\bar{x}, \omega_i\right) + \epsilon_n e_i(\bar{x}) \nsubseteq -\mathrm{int} C_i(\bar{x}), \quad \forall \omega_i \in \Gamma_i\left(\bar{x}, p^n\right).$$

Since, for each $i \in I$, the set-valued mappings $F_i : H \times K \times H_i \to 2^{Z_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ are continuous, there exist continuous selections f_i and ψ_i of F_i and Ψ_i, respectively, such that

$$f_i\left(\bar{x}, \bar{y}^n, \omega_i\right) + \psi_i\left(\bar{x}, \omega_i\right) + \epsilon_n e_i(\bar{x}) \in W_i(\bar{x}), \quad \forall \omega_i \in \Gamma_i\left(\bar{x}, p^n\right). \tag{4.7}$$

Since the mappings e_i, F_i, Ψ_i are continuous, the set-valued mapping T_i is upper semicontinuous and compact-valued and $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is lower semicontinuous and closed, taking the limit in (4.6) and (4.7), we can conclude that

$$d_i\left(\bar{x}_i, \Gamma_i\left(\bar{x}, p^*\right)\right) = 0 \tag{4.8}$$

and there exist a subsequence $\{\bar{y}_i^{n_k}\}$ of $\{\bar{y}_i^n\}$ and $\bar{y}_i \in T_i(\bar{x}, p^*)$ such that $\bar{y}_i^{n_k} \to \bar{y}_i$ and $f_i(\bar{x}, \bar{y}, \omega_i) \in F_i(\bar{x}, \bar{y}, \omega_i)$. Hence we have

$$f_i(\bar{x}, \bar{y}, \omega_i) + \psi_i(\bar{x}, \omega_i) \in W_i(\bar{x}), \quad \forall \omega_i \in \Gamma_i(\bar{x}, p^*),$$

which yields that

$$F_i(\bar{x}, \bar{y}, \omega_i) + \Psi_i(\bar{x}, \omega_i) \nsubseteq -\mathrm{int} C_i(\bar{x}), \quad \forall \omega_i \in \Gamma_i(\bar{x}, p^*).$$

Therefore, $\bar{x} \in S$, that is, $\bigcap_{\epsilon>0} \Omega_2(\epsilon) \subseteq S$.

Similarly, we know that $\Omega_1(\epsilon)$ is closed for all $\epsilon \geq 0$ and $S = \bigcap_{\epsilon>0} \Omega_1(\epsilon)$. This completes the proof. □

Remark 4.4 From Lemma 4.1, the solution set S of (SSVQEP) is closed.

In the sequel, we always assume that $x^* \in H$ is a fixed solution of (SSVQEP). Define

$$\theta_j(\epsilon) = \sup d\left(x^*, \Omega_j(\epsilon)\right), \quad \forall \epsilon > 0, \ j = 1, 2.$$

It is easy to see that $\theta_j(\epsilon)$ is the radius of the smallest closed ball centered at x^* containing $\Omega_j(\epsilon)$ for $j = 1, 2$.

Theorem 4.1 *(SSVQEP) is the type II LP well-posed by perturbations if and only if* $\theta_2(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Proof Let (SSVQEP) be the type II LP well-posed by perturbations. We know that $S = \{x^*\}$. Suppose that $\theta_2(\epsilon) \nrightarrow 0$ as $\epsilon \to 0$. Then there exist $\sigma > 0$ and a sequence $\{\epsilon_n\}$ of nonnegative real numbers with $\epsilon_n \to 0$ such that $\theta_2(\epsilon_n) > \sigma$. That is, there exists $x^n \in \Omega_2(\epsilon_n)$ such that

$$d(x^*, x^n) = \|x^n - x^*\| \geq \sigma. \tag{4.9}$$

Again, from $x^n \in \Omega_2(\epsilon_n)$, there exists $p^n \in B(p^*, \epsilon_n)$ such that, for each $i \in I$,

$$d_i\left(x_i^n, \Gamma_i\left(x^n, p^n\right)\right) \leq \epsilon_n$$

and there exists $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i\left(x^n, y^n, \omega_i\right)+\Psi_i\left(x^n, \omega_i\right)+\epsilon_n e_i\left(x^n\right) \nsubseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right), \ n \in N.$$

Therefore, $\{x^n\}$ is a type II LP approximating solution sequence corresponding to $\{p^n\}$ for (SSVQEP) whenever $p^n \to p^*$ and so

$$\|x^n - x^*\| \to 0,$$

which contradicts (4.9).

Conversely, let $\theta_2(\epsilon) \to 0$ as $\epsilon \to 0$. Clearly, $S = \{x^*\}$. If not, take $\hat{x} \in S$ arbitrarily and let $\hat{x} \neq x^*$. Then $\hat{x} \in \Omega_2(\epsilon)$ and so $\theta_2(\epsilon) \geq \|\hat{x} - x^*\| > 0$, which is a contradiction. On the other hand, let $\{p^n\} \subset \bigwedge$ with $p^n \to p^*$. Let $\{x^n\}$ be a type II LP approximating solution sequence corresponding to $\{p^n\}$ for (SSVQEP). Then there exist a sequence $\{\epsilon_n\}$ of nonnegative real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$d_i\left(x_i^n, \Gamma_i\left(x^n, p^n\right)\right) \leq \epsilon_n$$

and

$$F_i\left(x^n, y^n, \omega_i\right)+\Psi_i\left(x^n, \omega_i\right)+\epsilon_n e_i\left(x^n\right) \nsubseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right), \ n \in N.$$

Set $\sigma_n = \|p^n - p^*\|$ and $\tilde{\epsilon}_n = \max\{\sigma_n, \epsilon_n\}$. It follows that $\tilde{\epsilon}_n \to 0$ and $x^n \in \Omega_2(\tilde{\epsilon}_n)$. From the definition of θ_2, we have

$$\theta_2(\tilde{\epsilon}_n) \geq \|x^n - x^*\|$$

and so $\|x^n - x^*\| \to 0$. Therefore, (SSVQEP) is type II LP well-posed by perturbations. This completes the proof. □

Theorem 4.2 *(SSVQEP) is the type I LP well-posed by perturbations if and only if* $\theta_1(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Proof The proof is similar to the proof of Theorem 4.1 and so is omitted here. □

Theorem 4.3 *(SSVQEP) is the generalized type I LP well-posed by perturbations if and only if S is nonempty compact and* $\mathscr{H}(\Omega_1(\epsilon), S) \to 0$ *as* $\epsilon \to 0$.

Proof Let (SSVQEP) be the generalized type I LP well-posed by perturbations. By Remark 4.2, S is nonempty and compact. Suppose that $\mathscr{H}(\Omega_1(\epsilon), S) \not\to 0$ as $\epsilon \to 0$, that is, $e(\Omega_1(\epsilon), S) \not\to 0$ as $\epsilon \to 0$. Indeed, $e(S, \Omega_1(\epsilon)) = 0$ since $S \subseteq \Omega_1(\epsilon)$. Moreover, we have

$$\mathscr{H}(\Omega_1(\epsilon), S) = \max\{e(\Omega_1(\epsilon), S), e(S, \Omega_1(\epsilon))\} = e(\Omega_1(\epsilon), S).$$

Therefore, there exist $\sigma > 0$, $\epsilon_n > 0$ with $\epsilon_n \to 0$ and $x^n \in \Omega_1(\epsilon_n)$ such that

$$d(x^n, S) > \sigma. \tag{4.10}$$

Since $x^n \in \Omega_1(\epsilon_n)$, there exists $p^n \in B(p^*, \epsilon_n)$ such that, for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exists $y_i^n \in T_i(x^n, p^n)$ satisfy

$$F_i\left(x^n, y^n, \omega_i\right) + \Psi_i\left(x^n, \omega_i\right) + \epsilon_n e_i\left(x^n\right) \nsubseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right).$$

It follows that $p^n \to p^*$ and $\{x^n\}$ is a type I LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP). By the generalized type I LP well-posed by perturbations of (SSVQEP), there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ which converges strongly to some point of S. In other word, $d(x^{n_k}, S) \to 0$ as $k \to \infty$, which contradicts (4.10).

Conversely, assume that S is nonempty compact and $\mathscr{H}(\Omega_1(\epsilon), S) \to 0$ as $\epsilon \to 0$. Let $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^*$ and $\{x^n\}$ be a type I LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP). Then $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence $\{\epsilon_n\}$ of nonnegative real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i\left(x^n, y^n, \omega_i\right) + \Psi_i\left(x^n, \omega_i\right) + \epsilon_n e_i\left(x^n\right) \nsubseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right), \ n \in N.$$

Consequently, $\{x^n\} \subseteq \Omega_1(\epsilon_n)$. Take into account of $\mathscr{H}(\Omega_1(\epsilon), S) \to 0$ as $\epsilon \to 0$, we obtain

$$d(x^n, S) \leq e(\Omega_1(\epsilon), S) = \mathscr{H}(\Omega_1(\epsilon), S) \to 0.$$

Since S is nonempty, then there exists a sequence $\{\bar{x}^n\} \subseteq S$ such that

$$d\left(x^n, \bar{x}^n\right) \to 0.$$

From the compactness of S, it follows that there exists a subsequence $\{\bar{x}^{n_k}\}$ of $\{\bar{x}^n\}$ which converges strongly to some point $\bar{x} \in S$. Then there exists an associated subsequence $\{x^{n_k}\}$ of $\{x^n\}$ such that $x^{n_k} \to \bar{x}$. Therefore, (SSVQEP) is generalized type I LP well-posed by perturbations. This completes the proof. □

Theorem 4.4 *[SSVQEP] is the generalized type II LP well-posed by perturbations if and only if S is nonempty compact and* $\mathscr{H}(\Omega_2(\epsilon), S) \to 0$ *as* $\epsilon \to 0$.

Proof The proof is similar to the proof of Theorem 4.3 and so is omitted here. □

The following examples illustrate that the compactness of the solution set S in Theorems 4.3 and 4.4 is necessary:

Example 4.1 Let I be a single set, $\bigwedge = (-1, 1), X = Y = Z = R = (-\infty, +\infty), C(x) = R^+ = [0, +\infty)$ for all $x \in X, H = K = R^+ = [0, +\infty)$ and $T(x, p) = \Gamma(x, p) = [0, x], F(x, y, z) = y + z - x, \Psi(x, z) = 2(x - z)$ for all $x, y, z \in X$ and $p \in \bigwedge$. Clearly, the solution set $S = H$ and so it is not compact. Moreover, $\mathscr{H}(\Omega_j(\epsilon), S) \to 0$ as $\epsilon \to 0$ since $S \subseteq \Omega_j(\epsilon) \subseteq H$ for any $\epsilon > 0$. Let $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^* \in \bigwedge$. However, the sequence $\{n\}$ is a type I (resp., type II) LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP), but the sequence $\{n\}$ has no convergent subsequence. Therefore, (SSVQEP) is not the generalized type I (resp., type II) LP well-posed by perturbations.

Example 4.2 Let I be a single set, $\bigwedge = (-1, 1), X = Y = Z = R = (-\infty, +\infty), C(x) = R^+ = [0, +\infty)$ for all $x \in X, H = K = R^+ = [0, +\infty)$ and $T(x, p) = [0, x + p + 1], \Gamma(x, p) = [0, x], F(x, y, z) = [-(x + y - z), 0], \Psi(x, z) = [x - z, 2x - z]$ for all $x, y, z \in X$ and $p \in \bigwedge$. Simple computation allows that the solution set $S = H$ and so it is not compact. Moreover, $\mathscr{H}(\Omega_j(\epsilon), S) \to 0$ as $\epsilon \to 0$ since $S \subseteq \Omega_j(\epsilon) \subseteq H$ for any $\epsilon > 0$. Let $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^* \in \bigwedge$. However, the sequence $\{n\}$ is a type I (resp., type II) LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP), but the sequence $\{n\}$ has no convergent subsequence. Therefore, (SSVQEP) is not the generalized type I (resp., type II) LP well-posed by perturbations.

Now, we give the Furi-Vignoli type characterization Furi and Vignoli (1970) of the generalized type I (type II) LP well-posedness by perturbations for (SSVQEP) by using Kuratowski measure of noncompactness instead of the diameter. Since (SSVQEP) has more than one solutions, the diameters of the approximating solution sets $\Omega_1(\epsilon)$ and $\Omega_2(\epsilon)$ do not tend to zero, respectively.

Theorem 4.5 *Assume that all the conditions of Lemma 4.1 are satisfied. Then we have the following:*

(1) *(SSVQEP) is the generalized type I LP well-posed by perturbations if and only if*

$$\Omega_1(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \quad \lim_{\epsilon \to 0} \mathscr{M}(\Omega_1(\epsilon)) = 0.$$

(2) *(SSVQEP) is the generalized type II LP well-posed by perturbations if and only if*

$$\Omega_2(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0, \quad \lim_{\epsilon \to 0} \mathscr{M}(\Omega_2(\epsilon)) = 0.$$

Proof By Lemma 4.1, for each $j \in \{1, 2\}$, $\Omega_j(\epsilon)$ is closed for all $\epsilon \geq 0$ and $S = \bigcap_{\epsilon > 0} \Omega_j(\epsilon)$.

(1) Suppose that (SSVQEP) is the generalized type II LP well-posed by perturbations. It follows from Theorem 4.3 that S is nonempty compact and $\mathscr{H}(\Omega_1(\epsilon), S) \to 0$ as $\epsilon \to 0$. Since $S \subseteq \Omega_1(\epsilon)$ for all $\epsilon > 0$, we have $\Omega_1(\epsilon) \neq \emptyset$ for all $\epsilon > 0$. In the light of Remark 2.1, one has

$$\mathscr{M}(\Omega_1(\epsilon)) \leq 2\mathscr{H}(\Omega_1(\epsilon), S) = 2e(\Omega_1(\epsilon), S).$$

Therefore, $\mathscr{M}(\Omega_1(\epsilon)) \to 0$ as $\epsilon \to 0$, that is, $\lim_{\epsilon\to 0} \mathscr{M}(\Omega_1(\epsilon)) = 0$.

Conversely, let $\Omega_1(\epsilon) \neq \emptyset$ for all $\epsilon > 0$ and $\lim_{\epsilon\to 0} \mathscr{M}(\Omega_1(\epsilon)) = 0$. From the definition of Ω_1, we get

$$\Omega_1(\tilde{\epsilon}) \subseteq \Omega_1(\hat{\epsilon}), \quad \forall \tilde{\epsilon}, \hat{\epsilon} \in R^+ \setminus \{0\}\ (\tilde{\epsilon} \leq \hat{\epsilon}),$$

that is, Ω_1 is increasing on $R^+ \setminus \{0\}$. Again, from $S = \bigcap_{\epsilon>0} \Omega_j(\epsilon)$ and the Kuratowski theorem (see Kuratowski 1968), one has

$$\mathscr{H}(\Omega_1(\epsilon), S) \to 0 \quad (\epsilon \to 0)$$

and S is nonempty and compact. Thus, in view of Theorem 4.3, (SSVQEP) is the generalized type II LP well-posed by perturbations.

(2) The proof is similar to that of (1) and so is omitted. This completes the proof.

□

It is well known that the well-posedness of a optimization problem is equivalent to the existence and uniqueness of its solutions. In Hu et al. (2010b), also obtained the relations among the well-posedness, the existence and uniqueness of solutions for system of equilibrium problems.

Next, we establish analogous results for the type I LP well-posedness (resp., generalized type I LP well-posedness, type II LP well-posedness and generalized type II LP well-posedness) by perturbations of (SSVQEP).

Theorem 4.6 *Assume that all the conditions of Lemma 4.1 are satisfied and X is a finite dimensional space. If $\Omega_1(\tilde{\epsilon})$ and $\Omega_2(\tilde{\epsilon})$ are nonempty bounded for $\tilde{\epsilon} > 0$. Then we have the following:*

(1) *(SSVQEP) is the type I LP well-posed by perturbations if and only if it has a unique solution.*
(2) *(SSVQEP) is the type II LP well-posed by perturbations if and only if it has a unique solution.*

Proof (1) It immediately follows from the definition of the type I LP well-posed by perturbations of (SSVQEP) that (SSVQEP) has a unique solution.

Conversely, suppose that $S = \{x^*\}$. For any $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^*$, let $\{x^n\}$ be any type I LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP).

Then, for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i\left(x^n, y^n, \omega_i\right)+\Psi_i\left(x^n, \omega_i\right)+\epsilon_n e_i\left(x^n\right) \nsubseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right),\ n \in N.$$

Let $\tilde{\epsilon}_n = \max\{\epsilon_n, \|p^n - p^*\|\}$. Then $x^n \in \Omega_1(\tilde{\epsilon}_n)$. Since $\Omega_1(\tilde{\epsilon})$ is nonempty bounded for $\tilde{\epsilon} > 0$, there exists $\tilde{n} \in N$ such that $\{x^n\} \subseteq \Omega_1(\tilde{\epsilon}_n) \subseteq \Omega_1(\tilde{\epsilon})$ for all $n \geq \tilde{n}$. So, $\{x^n\}$ is bounded. Let $\{x^{n_k}\}$ be any subsequence of $\{x^n\}$ with $x^{n_k} \to \bar{x}$. By the similar proof of Step 1 in Lemma 4.1, we obtain that $\bar{x} \in S$. Again, from $S = \{x^*\}$, we have $\bar{x} = x^*$. Moreover, x^n converges strongly to x^*. Therefore, (SSVQEP) is the type I LP well-posed by perturbations.

(2) The proof of (2) is similar to that of (1) and so is omitted. This completes the proof. □

Example 4.3 Let I be a single set, $\bigwedge = (-1, 1)$, $X = Y = Z = R = (-\infty, +\infty)$, $C(x) = R^+ = [0, +\infty)$ for all $x \in X$, $H = K = [-1, 0]$ and $T(x, p) = \{-1\}$, $\Gamma(x, p) = [x, 0]$, $F(x, y, z) = -(x - y - z)$, $\Psi(x, z) = -z$ for all $x, y, z \in X$ and $p \in \bigwedge$. It is easy to verify that the solution set $S = \{-1\}$, there exists $\tilde{\epsilon} = \frac{1-|p|}{3}$ such that $\Omega_1(\frac{1-|p|}{3}) = \Omega_2(\frac{1-|p|}{3}) = [-1, -\frac{2+|p|}{3}]$ is nonempty bounded and the assumptions of Theorem 4.6 are satisfied. So, from Theorem 4.6, it follows that (SSVQEP) is the type I (resp., type II) LP well-posed by perturbations.

Theorem 4.7 *Assume that all the conditions of Theorem 4.6 are satisfied. If $\Omega_1(\tilde{\epsilon})$ (resp., $\Omega_2(\tilde{\epsilon})$) is nonempty bounded for some $\tilde{\epsilon} > 0$, then (SSVQEP) is the generalized type I (resp., generalized type II) LP well-posed by perturbations.*

Proof We first prove that (SSVQEP) is the generalized type I LP well-posed by perturbations. For any $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^*$, let $\{x^n\}$ be any type I LP approximating solution sequence corresponding to $\{p^n\}$ of (SSVQEP). Then, for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence $\{\epsilon_n\}$ of positive real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$F_i\left(x^n, y^n, \omega_i\right)+\Psi_i\left(x^n, \omega_i\right)+\epsilon_n e_i\left(x^n\right) \nsubseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right),\ n \in N.$$

Put $\iota_n = \max\{\epsilon_n, \|p^n - p^*\|\}$. Then $\iota_n \to 0$ and $x^n \in \Omega_1(\iota_n)$ for each $n \in N$. Furthermore, we have $x^n \in \Omega_1(\tilde{\epsilon})$ for all sufficiently large n. From the boundness of $\Omega_1(\tilde{\epsilon})$, it follows that there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ with $x^{n_k} \to \bar{x}$. As in the proof of Theorem 4.6, one can conclude that $\bar{x} \in S$ and so $S \neq \emptyset$. Therefore, (SSVQEP) is the generalized type I LP well-posed by perturbations.

Similarly, we can derive that (SSVQEP) is the generalized type II LP well-posed by perturbations. This completes the proof. □

Remark 4.5 (1) Theorem 4.7 means that the generalized type I (resp., generalized type II) LP well-posed by perturbations of (SSVQEP) is equivalent to the existence of its solutions under some suitable conditions.

(2) In Theorem 4.7, the boundness of $\Omega_j(\tilde{\epsilon})$ for $j = 1, 2$ is necessary for some $\tilde{\epsilon} > 0$.

Example 4.4 Let I be a single set, $\bigwedge = (-1, 1), X = Y = Z = R = (-\infty, +\infty), C(x) = R^+ = [0, +\infty)$ for all $x \in X, H = K = R^+ = [0, +\infty)$ and $T(x, p) = \Gamma(x, p) = [0, x], F(x, y, z) = -(x + y - z), \Psi(x, z) = 2(x - z)$ for all $x, y, z \in X$ and $p \in \bigwedge$. It is easy to verify that the assumptions of Theorem 4.7 are satisfied. However, $\Omega_j(\epsilon)$ for $j = 1, 2$ are unbounded for all $\epsilon > 0$ since the solution set $S = H$ is unbounded. So, from Theorem 4.7, (SSVQEP) is not the generalized type I (resp., type II) LP well-posed by perturbations.

5 Links with Levitin–Polyak well-posedness for minimization problems with constraints

In this section, by virtue of the nonlinear scalarization function introduced by Chen et al. (2005a), we present a parametric gap function g for (PSSVQEP) and the continuity of the parametric gap function and establish the equivalence between the type I (resp., type II, generalized type I and generalized type II) Levitin–Polyak well-posedness by perturbations of (SSVQEP) and the corresponding minimization problem with functional constraints under quite mild assumptions.

Definition 5.1 A mapping $g : H \times \Lambda \to R \bigcup\{+\infty\}$ is called the *parametric gap function* for (PSSVQEP) if

(i) $g(x, p) \geq 0$ for all $(x, p) \in H \times \Lambda$;
(ii) $g(x^*, p^*) = 0$ for some $(x^*, p^*) \in H \times \Lambda$ if and only if $x^* \in S(p^*)$.

Remark 5.1 The definition of the parametric gap function for (PSSVQEP) is different from that of Peng et al. (2012), which do not involve any other functions.

For each parametric $p \in \Lambda$, we define the following function $g : H \times \Lambda \to R \bigcup\{+\infty\}$ by

$$g(x, p) =: \max_{i \in I} \min_{y_i \in T_i(x,p)} \max_{\omega_i \in \Gamma_i(x,p)} \min_{f_i \in F_i(x,y,\omega_i), \psi_i \in \Psi_i(x,\omega_i)} -\xi_{e_i}(x, f_i + \psi_i) \tag{5.1}$$

for all $(x, p) \in H \times \Lambda$.

In this section, we always suppose that I is an compact index set, for each $i \in I$, $F_i : H \times K \times H_i \to 2^{Z_i}, \Gamma_i : H \times \Lambda \to 2^{H_i}, T_i : H \times \Lambda \to 2^{K_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ are nonempty compact-valued mappings, where $H = \prod_{i \in I} H_i$ and $K = \prod_{i \in I} K_i$. Then the function g is well-defined.

It is well known that the gap functions are widely applied in optimization problems, equation problems, variational inequalities problems and others problems. The minimization of the gap function is a effectively approach for solving variational inequalities and equilibrium problems. Many authors have studied the gap functions and applied to construct some novel algorithms for variational inequalities and equilibrium problems (see, for example, Chen et al. 2000; Li et al. 2006 and the references therein). From the computational point of view, the real-valued gap functions may be more useful.

Lemma 5.1 *Let Λ be a metric space, I be an compact index set and, for each $i \in I$, Z_i be topological vector space, H_i and K_i be nonempty closed convex subsets of locally convex Hausdorff topological vector spaces X_i and Y_i, respectively. Let $C_i : H \to 2^{Z_i}$ be a set-valued mappings such that, for each $x \in H$, $C_i(x)$ is a proper closed convex and pointed cone in Z_i with $intC_i(x) \neq \emptyset$ and $e_i : H \to Z_i$ be a vector-valued mapping with $e_i(x) \in intC(x)$ for all $x \in H$. Assume that, for each $p \in \Lambda$ and $i \in I$, $x_i \in \Gamma_i(x, p)$ and there exists $y_i \in T_i(x, p)$ such that*

$$F_i(x, y, x_i) + \Psi_i(x, x_i) \subseteq -\partial C_i(x). \tag{5.2}$$

Then the function $g(x, p)$ is a parametric gap function of (PSSVQEP) with respect to the parametric p.

Proof Taking $(x, p) \in H \times \Lambda$ arbitrarily. Suppose that $g(x, p) < 0$. From the definition of g, for each $i \in I$, there exists $y_i \in T_i(x, p)$ such that

$$\min_{f_i \in F_i(x,y,\omega_i), \psi_i \in \Psi_i(x,\omega_i)} -\xi_{e_i}(x, f_i + \psi_i) < 0, \quad \forall \omega_i \in \Gamma_i(x, p).$$

This implies that there exist $\bar{f_i} \in F_i(x, y, x_i)$ and $\bar{\psi_i} \in \Psi_i(x, x_i)$ such that

$$-\xi_{e_i}(x, \bar{f_i} + \bar{\psi_i}) < 0.$$

By Lemma 2.1 (4), we have $\bar{f_i} + \bar{\psi_i} \notin -C_i(x)$, which contradicts (5.2). Therefore, $g(x, p) \geq 0$ for all $(x, p) \in H \times \Lambda$.

Secondly, we show that $(x_0, p_0) \in H \times \Lambda$ satisfying $g(x_0, p_0) = 0$ if and only if $x_0 \in S(p_0)$.

In fact, suppose that $g(x_0, p_0) = 0$ for some $(x_0, p_0) \in H \times \Lambda$. Since I is an compact index set and, for each $i \in I$, $F_i : H \times K \times H_i \to 2^{Z_i}$, $\Gamma_i : H \times \Lambda \to 2^{H_i}$, $T_i : H \times \Lambda \to 2^{K_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ are nonempty compact-valued, it follows that, for each $i \in I$, there exists $y_{0i} \in T_i(x_0, p_0)$ such that

$$\min_{f_i \in F_i(x_0,y_0,\omega_i), \psi_i \in \Psi_i(x_0,\omega_i)} -\xi_{e_i}(x_0, f_i + \psi_i) \leq 0, \quad \forall \omega_i \in \Gamma_i(x_0, p_0).$$

Furthermore, for any $\omega_i \in \Gamma_i(x_0, p_0)$, there exist $\tilde{f_i} \in F_i(x_0, y_0, \omega_i)$ and $\tilde{\psi_i} \in \Psi_i(x_0, \omega_i)$ such that $-\xi_{e_i}(x_0, \tilde{f_i} + \tilde{\psi_i}) \leq 0$. Thus, from this and Lemma 2.1, it follows that

$$\tilde{f_i} + \tilde{\psi_i} \notin -\text{int}C_i(x_0).$$

Therefore, for each $i \in I$, $x_{0i} \in \Gamma_i(x_0, p_0)$ and there exists $y_{0i} \in T_i(x_0, p_0)$ such that

$$F_i(x_0, y_0, \omega_i) + \Psi_i(x_0, \omega_i) \nsubseteq -\text{int}C_i(x_0), \quad \forall \omega_i \in \Gamma_i(x_0, p_0),$$

that is, $x_0 \in S(p_0)$.

Conversely, assume that $x_0 \in S(p_0)$ for some $(x_0, p_0) \in H \times \Lambda$. Then $g(x_0, p_0) \geq 0$. Let us show that $g(x_0, p_0) = 0$. Suppose that $g(x_0, p_0) > 0$. By the definition of g, one can obtain

$$g(x_0, p_0) =: \max_{i \in I} \min_{y_{0i} \in T_i(x_0, p_0)} \max_{\omega_i \in \Gamma_i(x_0, p_0)} \min_{f_i \in F_i(x_0, y_0, \omega_i), \psi_i \in \Psi_i(x_0, \omega_i)} -\xi_{e_i}(x_0, f_i + \psi_i) > 0,$$

that is, for some $i' \in I$, $y_{0i'} \in T_{i'}(x_0, p_0)$ and there exists $\omega_{0i'} \in \Gamma_{i'}(x_0, p_0)$ such that

$$\xi_{e_{i'}}(x_0, f_{i'} + \psi_{i'}) < 0, \quad \forall f_{i'} \in F_{i'}(x_0, y_0, \omega_{0i'}), \; \psi_{i'} \in \Psi_{i'}(x_0, \omega_{0i'}).$$

Thus, from this and Lemma 2.1 (1), it follows that

$$f_{i'} + \psi_{i'} \in -\text{int}C_{i'}(x_0), \quad \forall f_{i'} \in F_{i'}(x_0, y_0, \omega_{0i'}), \; \psi_{i'} \in \Psi_{i'}(x_0, \omega_{0i'}).$$

Thus we have

$$F_{i'}(x_0, y_0, \omega_{0i'}) + \Psi_{i'}(x_0, \omega_{0i'}) \subseteq -\text{int}C_{i'}(x_0),$$

which contradicts $x_0 \in S(p_0)$ and so $g(x_0, p_0) = 0$. Therefore, the function $g(x, p)$ is a parametric gap function of (PSSVQEP) with respect to the parametric p. This completes the proof. □

By the following example, we can show that the nonlinear scalarization function is computable (see, Chen et al. 1999; Chen and Yang 2002; Li and Li 2009) and the assumption of Lemma 5.1 is satisfied.

Example 5.1 Let I be a singleton, $\Lambda = (0, 1)$, $X = Y = K = R = (-\infty, +\infty)$, $Z = R^3$, $H = R_+ = [0, +\infty)$, $C(x) = R^3_+$ and $e(x) = (1, 1, 1)$ for all $x \in X$. For each $p \in \Lambda$ and $x, \omega \in H$, let $\Gamma(x, p) = [x, x + p]$, $T(x, p) = [x - p, x]$, $F(x, y, \omega) = \{0\} \times \{-\omega + x\} \times [0, y + 1]$ and $\Psi(x, \omega) = \{(0, -2\omega + x, -x - 1)\}$. Set $S = \{z = (z_1, z_2, z_3) \in R^3 : h_i(z) \leq 0, h_i \in Z^* \setminus \{0\}, i = 1, 2, 3\}$, where $h_i(z) = -z_i, i = 1, 2, 3$ and Z^* is the dual space of Z. Clearly, $S = C(x) = R^3_+$. It is easy to see that, for each $p \in \Lambda$, $x \in \Gamma(x, p)$ and $y \in T(x, p)$ such that

$$\begin{aligned} F(x, y, x) + \Psi(x, x) &= \{0\} \times \{0\} \times [0, y + 1] + (0, -x, -x - 1) \\ &= \{0\} \times \{-x\} \times [-x - 1, y - x]. \end{aligned}$$

So, there exists $y = x$ such that

$$F(x, y, x) + \Psi(x, x) = \{(0, -x, \iota) : \iota \in [-x - 1, 0]\} \subseteq -\partial C(x) = -\partial R^3_+.$$

Simple computation shows that, for each $p \in \Lambda$, the solution set $S(p) = H = R_+$. By Corollary 2.5 of Chen et al. (1999, p. 244), one has

$$\begin{aligned} &\xi_e(x, F(x, y, \omega) + \Psi(x, \omega)) \\ &\quad = \max\{h_1(f + \psi), h_2(f + \psi), h_3(f + \psi) : f \in F(x, y, \omega), \psi \in \Psi(x, \omega)\} \\ &\quad = \max\{0, 2x - 3\omega, \iota : \iota \in [-x - 1, y - x]\}. \end{aligned}$$

Therefore,

$$\min_{f_i \in F_i(x,y,\omega_i), \psi_i \in \Psi_i(x,\omega_i)} -\xi_{e_i}(x, f_i + \psi_i) = \min\{0, 3\omega - 2x, \iota : \iota \in [x-y, x+1]\}.$$

Moreover, we have

$$g(x,p) = \min_{y \in T(x,p)} \max_{\omega \in \Gamma(x,p)} \min\{0, 3\omega - 2x, \iota : \iota \in [x-y, x+1]\}.$$

It follows from the proof of Lemma 5.1 that $g(x, p)$ is a parametric gap function of (PSSVQEP) with respect to the parametric p.

Now we give an example to illustrate that the gap function defined by (5.1) is applicable, but the gap function defined by Peng et al. (2012, p. 876, (20)) fails.

Example 5.2 Let X, Y, K, Z be the same as Example 5.1, $H = [0, 1]$ and let $C(x) = R^3_+$ and $e(x) = (1, 1, 1)$ for all $x \in X$. For each $x, \omega \in H$, let $\Gamma(x) = [x, 1]$, $T(x) = [x-1, x]$, $F(x, y, \omega) = (0, -\omega + x, y)$ and $\Psi(x, \omega) = (0, -2\omega + x, -x)$. Set $S = \{z = (z_1, z_2, z_3) \in R^3 : h_i(z) \le 0, h_i \in Z^* \setminus \{0\}, i = 1, 2, 3\}$, where $h_i(z) = -z_i$, $i = 1, 2, 3$ and Z^* is the dual space of Z. Clearly, $S = C(x) = R^3_+$. For the following generalized vector equilibrium problem (GVEP) studied by Peng et al. (2012): finding $x \in H$, such that $x \in \Gamma(x)$ and there exists $y \in T(x)$ satisfying

$$F(x, y, \omega) \notin -\text{int}C(x), \quad \forall \omega \in \Gamma(x).$$

It is easy to see that the set of solutions to (GVEP) is H. However, the function defined by Peng et al. (2012, p. 876, (20)):

$$\begin{aligned}\phi(x) &= \inf_{y \in T(x)} \max_{\omega \in \Gamma(x)} -\xi_e(x, F(x, y, \omega)) \\ &= \inf_{y \in T(x)} \max_{\omega \in \Gamma(x)} \min\{0, \omega - x, -y\} \\ &< 0, \quad \forall x \in H \setminus \{0\}.\end{aligned}$$

That is, for any solution $x \in H \setminus \{0\}$ of (GVEP), $\phi(x) < 0$. So, the function ϕ fails to be the gap function of (GVEP).

On the other hand, (GVEP) is equivalent to the following generalized vector quasi-equilibrium problem (GVQEP):

$$F(x, y, \omega) + \Psi(x, \omega) = (0, 2x - 3\omega, y - x) \notin -\text{int}C(x), \quad \forall \omega \in \Gamma(x).$$

In fact, let $p = 1$, for any $x \in H$, $x \in \Gamma(x, p) = [x, p] = \Gamma(x)$ and there exists $y = x - p \in T(x, p) = [x - p, x] = T(x)$ such that

$$F(x, y, \omega) + \Psi(x, \omega) = (0, 2x - 3\omega, -p) \notin -\text{int}C(x), \quad \forall \omega \in \Gamma(x, p).$$

This implies that $S(x, p) = H$. Similar to Example 5.1, we can show that

$$\xi_e(x, F(x, y, \omega) + \Psi(x, \omega)) = \max\{0, 2x - 3\omega, y - x\}$$

and

$$\min_{f_i \in F_i(x,y,\omega_i), \psi_i \in \Psi_i(x,\omega_i)} -\xi_{e_i}(x, f_i + \psi_i) = \min\{0, 3\omega - 2x, x - y\}.$$

Moreover, we have

$$g(x, p) = \min_{y \in T(x,p)} \max_{\omega \in \Gamma(x,p)} \min\{0, 3\omega - 2x, x - y\}, \quad x \in H.$$

It follows from the proof of Lemma 5.1 that $g(x, p)$ is a parametric gap function of (PSSVQEP) with respect to the parametric p. Therefore,

$$g(x) = \min_{y \in T(x)} \max_{\omega \in \Gamma(x)} \min\{0, 3\omega - 2x, x - y\}, \quad x \in H,$$

which can be viewed as a gap function of (GVEP) on H.

In the next, we investigate the continuity of the function g.

Lemma 5.2 *For each $i \in I$, let $C_i : H \to 2^{Z_i}$ be a set-valued mappings such that, for each $x \in H$, $C_i(x)$ is a proper closed convex and pointed cone in Z_i with $intC_i(x) \neq \emptyset$, $e_i : H \to Z_i$ be a continuous vector valued mapping with $e_i(x) \in intC_i(x)$ for any $x \in H$, the set-valued mappings $F_i : H \times K \times H_i \to 2^{Z_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ be continuous and compact-valued, the set-valued mapping $T_i : H \times \bigwedge \to 2^{K_i}$ be upper semicontinuous and compact-valued and $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ be lower semicontinuous and compact-valued. Assume that the mapping $\Theta_i(\cdot) = Z_i \setminus intC_i(\cdot)$ is upper semicontinuous on H. Then the function g defined by (5.1) is lower semicontinuous on $H \times \bigwedge$.*

Proof Since I is a compact index set, for each $i \in I$, the mappings F_i, Γ_i, T_i and Ψ_i are nonempty compact-valued and, from (5.1), one has

$$|g(x, p)| < +\infty, \quad \forall (x, p) \in H \times \bigwedge.$$

Now, we show that the function g is lower semicontinuous on $H \times \bigwedge$. Taking $\iota \in R$ arbitrarily. Assume that $\{(x^n, p^n)\} \subset H \times \bigwedge$ such that

$$\left(x^n, p^n\right) \to (x_0, p_0), \quad g\left(x^n, p^n\right) \leq \iota, \quad \forall n \in N,$$

that is, for each $n \in N$,

$$\max_{i \in I} \min_{y_i^n \in T_i(x^n,p^n)} \max_{\omega_i^n \in \Gamma_i(x^n,p^n)} \min_{f_i \in F_i(x^n,y^n,\omega_i^n), \psi_i \in \Psi_i(x^n,\omega_i^n)} -\xi_{e_i}(x^n, f_i + \psi_i) \leq \iota. \tag{5.3}$$

Since the mapping $\Theta_i(\cdot) = Z_i \setminus \text{int}C_i(\cdot)$ is upper semicontinuous on H, it follows from Lemma 2.2 (3) that $\xi_{e_i}(\cdot, \cdot)$ is upper semicontinuous on $H \times Z_i$ and so $-\xi_{e_i}(\cdot, \cdot)$ is lower semicontinuous on $H \times Z_i$. Note that the set-valued mapping $T_i : H \times \bigwedge \to 2^{K_i}$ is upper semicontinuous and compact-valued and $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is lower semicontinuous. From Remark 2.2, we can derive that, without loss of generality, for any $y_i^n \in T_i(x^n, p^n)$ and $\omega_{0i} \in \Gamma_i(x_0, p_0)$, there exist $y_{0i} \in T_i(x_0, p_0)$ and $\omega_i^n \in \Gamma_i(x^n, p^n)$ such that $y_i^n \to y_{0i}$ and $\omega_i^n \to \omega_{0i}$. Since the set-valued mappings $F_i : H \times K \times H_i \to 2^{Z_i}$ and $\Psi_i : H \times H_i \to 2^{Z_i}$ are continuous and compact-valued, we have

$$\begin{aligned} &\min_{f_i \in F_i(x_0, y_0, \omega_{0i}), \psi_i \in \Psi_i(x_0, \omega_{0i})} -\xi_{e_i}(x_0, f_i + \psi_i) \\ \leq &\min_{f_i \in F_i(x^n, y^n, \omega_i^n), \psi_i \in \Psi_i(x^n, \omega_i^n)} -\xi_{e_i}(x^n, f_i + \psi_i) \leq \iota, \quad \forall n \in N. \end{aligned}$$

This shows that

$$\max_{i \in I} \min_{y_{0i} \in T_i(x_0, p_0)} \max_{\omega_{0i} \in \Gamma_i(x_0, p_0)} \min_{f_i \in F_i(x_0, y_0, \omega_{0i}), \psi_i \in \Psi_i(x_0, \omega_{0i})} -\xi_{e_i}(x_0, f_i + \psi_i) \leq \iota,$$

i.e., $g(x_0, p_0) \leq \iota$. Thus the level set $\{(x, p) \in X \times \bigwedge : g(x, p) \leq \iota\}$ is closed-valued for all $\iota \in R$. Therefore, g is lower semicontinuous on $X \times \bigwedge$. This completes the proof. □

If the assumptions of Lemma 5.2 are enhanced, we can conclude that the function defined by (5.1) is continuous on $X \times \bigwedge$.

Lemma 5.3 *If the assumptions of Lemma 5.2 are satisfied, for each $i \in I$, $T_i : H \times \bigwedge \to 2^{K_i}$ is lower semicontinuous, $\Gamma_i : H \times \bigwedge \to 2^{H_i}$ is upper semicontinuous and compact-valued and the mapping $C_i(\cdot)$ is upper semicontinuous on H, then the function g defined by (5.1) is continuous on $H \times \bigwedge$.*

Proof By Lemma 5.2, we only need to prove that the function g defined by (5.1) is upper semicontinuous on $H \times \bigwedge$, that is, $-g$ is lower semicontinuous on $H \times \bigwedge$. The proof is similar to that of Lemma 5.2 and so it is omitted. This completes the proof. □

Example 5.3 Let us consider Example 5.1. By Example 5.1, for each $p \in (0, 1)$, $x \in H$, we have

$$g(x, p) = \min_{y \in T(x,p)} \max_{\omega \in \Gamma(x,p)} \min\{0, 3\omega - 2x, \iota : \iota \in [x - y, x + 1]\}.$$

It is easy to see that $g(x, p) = 0$ for $p \in (0, 1)$, $x \in H$, and so, the gap function g is lower semicontinuous (continuous) on $H \times \bigwedge$.

For any $p^* \in \bigwedge$, we consider the type I (resp., type II, generalized type I and generalized type II) LP well-posedness for a class of parametric optimization problem with constraints [for short, (P)]:

$$\begin{array}{ll} \min & g(x, p^*) \\ \text{subject to} & x_i \in \Gamma_i(x, p^*), \ \forall x \in H, \\ & y_i \in T_i(x, p^*), \ \forall i \in I, \end{array}$$

where I is an index set, for each $i \in I$, Z_i is a topological vector space, H_i and K_i are nonempty closed convex subsets of locally convex Hausdorff topological vector spaces X_i and Y_i, respectively, $\Gamma_i : H \times \Lambda \to 2^{H_i}$ and $T_i : H \times \Lambda \to 2^{K_i}$ are two set-valued mappings, where $H = \prod_{i \in I} H_i$ and $K = \prod_{i \in I} K_i$.

We denote the optimal set and optimal value of (P) by $\tilde{S}$ and $\tilde{\upsilon}$, respectively. In the following, we always assume that $\tilde{\upsilon} > -\infty$.

In the sequel, we recall some definitions on LP well-posedness for (P) which are needed in our results.

Definition 5.2 Let $\bigwedge$ be a metric space and $\{p^n\} \subseteq \bigwedge$ such that $p^n \to p^*$.

(1) A sequence $\{x^n\} \subseteq H$ is said to be the *type I LP minimizing sequence* corresponding to $\{p^n\}$ for (P) if, for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist a sequence $\{\epsilon_n\}$ of nonnegative real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$g\left(x^n, p^n\right) \leq \tilde{\upsilon} + \epsilon_n. \tag{5.4}$$

(2) A sequence $\{x^n\} \subseteq H$ is said to be the *type II LP minimizing sequence* corresponding to $\{p^n\}$ for (P) if, for each $i \in I$, there exist a sequence $\{\epsilon_n\}$ of nonnegative real numbers with $\epsilon_n \to 0$ and $y_i^n \in T_i(x^n, p^n)$ such that

$$d_i\left(x_i^n, \Gamma_i\left(x^n, p^n\right)\right) \leq \epsilon_n \tag{5.5}$$

and (5.4) hold.

In the following, we give the definitions of the LP well-posed by perturbations for (P) which are similar to Definition 4.2.

Definition 5.3 (1) (P) is called the *type I* (resp., *type II*) *LP well-posed by perturbations* if it has a unique solution and, for any $\{p^n\} \subset \bigwedge$ with $p^n \to p^*$, each type I (resp., type II) LP minimizing sequence corresponding to $\{p^n\}$ of (P) converges strongly to the unique solution.

(2) (P) is called the *generalized type I* (resp., *type II*) *LP well-posed by perturbations* if the solution set $\tilde{S}$ of (P) is nonempty and, for any $\{p^n\} \subset \bigwedge$ with $p^n \to p^*$, each type I (resp., type II) LP minimizing sequence corresponding to $\{p^n\}$ of (P) has a subsequence which converges strongly to some point of $\tilde{S}$.

Remark 5.2 Each type I LP well-posedness by perturbations for (P) is the type II (resp., generalized type I and generalized type II) LP well-posedness by perturbations for (P). Moreover, any generalized type I LP well-posedness by perturbations for (P) is the generalized type II LP well-posedness by perturbations for (P).

We explore the relationships between the type I (resp., type II, generalized type I and generalized type II) LP well-posedness of (P) and that of (SSVQEP).

Theorem 5.1 *Assume that the assumptions of Lemma 5.2 are satisfied. Then the following results hold:*

(1) *(SSVQEP) is the type I LP well-posedness by perturbations if and only if (P) is the type I LP well-posedness by perturbations with the function g defined by* (5.1).
(2) *(SSVQEP) is the type II LP well-posedness by perturbations if and only if (P) is the type II LP well-posedness by perturbations with the function g defined by* (5.1).
(3) *(SSVQEP) is the generalized type I LP well-posedness by perturbations if and only if (P) is the generalized type I LP well-posedness by perturbations with the function g defined by* (5.1).
(4) *(SSVQEP) is the generalized type II LP well-posedness by perturbations if and only if (P) is the generalized type II LP well-posedness by perturbations with the function g defined by* (5.1).

Proof We only need to prove (1). The proofs of (2), (3) and (4) are similar to that of (1), respectively, and so they are omitted. By Lemma 5.1, we know that $x^* \in S$ if and only if $x^* \in \tilde{S}$ corresponding to some parameters $p^* \in \bigwedge$ and $\tilde{\upsilon} = g(x^*, p^*) = 0$.

Suppose that (SSVQEP) is the type I LP well-posedness by perturbations. Let $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^*$ and $\{x^n\} \subseteq H$ be a type I LP approximating solution sequence corresponding to $\{p^n\}$ for (SSVQEP). Then, for each $i \in I$, $x_i^n \in \Gamma_i(x^n, p^n)$ and there exist $y_i^n \in T_i(x^n, p^n)$ and a sequence $\{\epsilon_n\}$ of nonnegative numbers with $\epsilon_n \to 0$ such that

$$F_i\left(x^n, y^n, \omega_i\right) + \Psi_i(x^n, \omega_i) + \epsilon_n e_i\left(x^n\right) \nsubseteq -\text{int}C_i\left(x^n\right), \quad \forall \omega_i \in \Gamma_i\left(x^n, p^n\right),\ n \in N.$$

This shows that there exist $f_i \in F_i(x^n, y^n, \omega_i)$ and $\psi \in \Psi_i\,(x^n, \omega_i)$ such that

$$f_i + \psi_i \notin -\epsilon_n e_i\left(x^n\right) - \text{int}C_i\left(x^n\right), \quad \forall n \in N.$$

Thus, from Lemma 2.1 (iii), one has $\xi_{e_i}(x^n, f_i + \psi_i) \geq -\epsilon_n$ for each $n \in N$, that is,

$$-\xi_{e_i}\left(x^n, f_i + \psi_i\right) \leq \epsilon_n, \quad \forall n \in N. \tag{5.6}$$

Therefore, from (5.6), it follows that, for each $n \in N$,

$$\max_{i \in I}\ \min_{y_i^n \in T_i(x^n, p^n)}\ \max_{\omega_i \in \Gamma_i(x^n, p^n)}\ \min_{f_i \in F_i(x^n, y^n, \omega_i), \psi_i \in \Psi_i(x^n, \omega_i)} -\xi_{e_i}(x^n, f_i + \psi_i) \leq \epsilon_n.$$

Thus, from this, we get $g(x^n, p^n) \leq \epsilon_n$ for each $n \in N$. Therefore, the sequence $\{x^n\}$ is a type I LP minimizing sequence corresponding to $\{p^n\}$ for (P). Since (SSVQEP) is the type I LP well-posedness by perturbations, (P) is the type I LP well-posedness by perturbations with the function g defined by (5.1).

Conversely, assume that (P) is the type I LP well-posedness by perturbations with the function g defined by (5.1). Let $\{p^n\} \subseteq \bigwedge$ with $p^n \to p^*$ and $\{x^n\} \subseteq H$ be a type I LP minimizing sequence corresponding to $\{p^n\}$ for (P). Then, for each

$i \in I, x_i^n \in \Gamma_i(x^n, p^n)$ and there exist $y_i^n \in T_i(x^n, p^n)$ and a sequence $\{\epsilon_n\}$ of nonnegative numbers with $\epsilon_n \to 0$ such that

$$g\left(x^n, p^n\right) \leq \tilde{\upsilon} + \epsilon_n = \epsilon_n, \quad \forall n \in N.$$

By the definition of g, for each $i \in I, \omega_i \in \Gamma_i(x^n, p^n)$, there exist $f_i \in F_i(x^n, y^n, \omega_i)$ and $\psi_i \in \Psi_i(x^n, \omega_i)$ such that

$$-\xi_{e_i}\left(x^n, f_i + \psi_i\right) \leq \epsilon_n, \quad \forall n \in N.$$

From Lemma 2.1 (3), one has $f_i + \psi_i \notin -\epsilon_n e_i(x^n) - \text{int}C_i(x^n)$ for each $n \in N$, that is,

$$F_i\left(x^n, y^n, \omega_i\right) + \Psi_i\left(x^n, \omega_i\right) + \epsilon_n e_i\left(x^n\right) \not\subseteq -\text{int}C_i\left(x^n\right), \quad \forall n \in N.$$

Therefore, the sequence $\{x^n\}$ is a type I LP approximating solution sequence corresponding to $\{p^n\}$ for (SSVQEP). From the type I LP well-posedness by perturbations with the function g defined by (5.1), it follows that (SSVQEP) is the type I LP well-posedness by perturbations. This completes the proof. □

6 Concluding remarks

In this paper, we firstly established the existence theorems of solutions for (PSSVQEP) and its dual problem (DPSSVQEP) under some suitable conditions.

Secondly, we introduce the notions of the type I (resp., type II, generalized type I and generalized type II) LP well-posedness by perturbations for (SSVQEP) in topological vector spaces. Some metric characterizations of these LP well-posedness by perturbations are derived. We also obtain the relationships among these LP well-posedness by perturbations and the existence and uniqueness of solution to (SSVQEP).

Finally, by virtue of the nonlinear scalarization function introduced by Chen et al. (2005a), a parametric gap function g for (PSSVQEP) is introduced, which is distinct from that of Peng et al. (2012). The continuity of the parametric gap function g is presented and then we establish the equivalence between these LP well-posedness by perturbations of (SSVQEP) and the corresponding minimization problem with functional constraints under quite mild assumptions. For further research, we may study the following problems:

(1) One can study the LP well-posed by perturbations for the dual problem of (SSVQEP), optimization problem with (SSVQEP) [resp., (PSSVQEP), (DSSVQEP) and (DPSSVQEP)] constraints, equilibrium problem with (SSVQEP) [resp., (PSSVQEP), (DSSVQEP) and (DPSSVQEP)] constraints.
(2) One also can study some metric characterizations and stability of approximating solution sets for the mentioned above problems.

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