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Strong convergence theorems for firmly nonexpansive-type mappings and equilibrium problems in Banach spaces

Jia-wei Chen [a], Zhongping Wan [a] & Yunzhi Zou [b]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P.R. China

[b] Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P.R. China

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Jia-wei Chen[a], Zhongping Wan[a]* and Yunzhi Zou[b]

[a]*School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, P.R. China;* [b]*Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P.R. China*

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In this article, we propose and investigate several iterative methods for approximating fixed points of a firmly nonexpansive-type mapping and for finding a common element of the set of fixed points of the firmly nonexpansive-type mapping and the set of solutions of an equilibrium problem in Banach space. By using the conception of generalized projection, strong convergence theorems for firmly nonexpansive-type mappings and equilibrium problems in Banach space are established under suitable assumptions, which extend and modify some known results in the literature.

Keywords: strong convergence theorem; firmly nonexpansive-type mapping; equilibrium problem; fixed point; generalized projection operator

AMS Subject Classifications: 47H09; 47J05; 47J25

1. Introduction

Let E be a real Banach space with the dual space E^*. The norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $C \subseteq E$ be a nonempty closed convex set. A mapping S on a subset C of E is called a nonexpansive mapping if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in C: Sx = x\}$. Let $M: E \to 2^{E^*}$ be a maximal monotone operator (see, e.g. [5,7,12,14,16]).

In [19], Reich proved a weak convergence theorem for finding a common asymptotic fixed point of a finite family of strongly nonexpansive mappings in a Banach space. Furthermore, he studied the proximal point algorithm for maximal monotone operators in a Banach space. By applying the conception of generalized projection, Kamimura et al. [12] introduced an iterative sequence for a maximal monotone operator, and proved the strong and weak convergence of the iterative sequence under different conditions. Moreover, they also explored the convex

*Corresponding author. Email: zpwan-whu@126.com

minimization problem and the variational inequality problem by the obtained results. In 2008, Li and Song [16] introduced the following algorithm:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}x_n)), \\ x_{n+1} = J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)J(y_n)), \quad n \in Z_+, \end{cases}$$

where $J_r = (J + rM)^{-1}J$, $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ and J is the duality mapping on E. They derived a strong convergence theorem and a weak convergence theorem under different conditions respectively, and gave an estimate of the convergence rate of the algorithm.

In Hilbert spaces, Nakajo and Takahashi [17] considered the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sx_n, \\ C_n = \{z \in C\colon \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ Q_n = \{z \in C\colon \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_n \cap Q_n}x, \quad n \in Z_+, \end{cases}$$

where C is a nonempty closed convex subset of a Hilbert space, P is metric projection operator, $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. They showed that $\{x_n\}$ converges strongly to $P_{F(S)}x_0$ by the hybrid method, and obtained a strong convergence theorem for a family of nonexpansive mappings in a Hilbert space.

Let f: $C \times C \to R$. Blum and Oettli [3] understood the equilibrium problem by finding $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

Denote the set of solutions of (1.1) by $EP(f)$. The equilibrium problem provided a very general formulation of variational problems such as:

(i) Minimization problem: find $x \in C$ such that $g(x) \leq g(y)$ for all $y \in C$, where g: $C \to R$ is a functional. In this case, we define $f(x, y) = g(y) - g(x)$ for all $x, y \in C$.
(ii) Variational inequality: find $x \in C$ such that $\langle G(x), y - x\rangle \geq 0$ for all $y \in C$, where G: $C \to E^*$ is a mapping. In this case, we define $f(x, y) = \langle G(x), y - x\rangle$ for all $x, y \in C$.

Considerable problems in physics, structural analysis, optimization, management science, economics and transportation equilibrium coincide to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problems (see, e.g. [3,4,6,21]). Recently, Takahashi and Zembayashi [22] proved the strong and weak convergence theorems for finding a common element of the set of solutions of the equilibrium problem (1.1) and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Ceng et al. [4] introduced the following algorithm:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}x_n))), \\ f(T_{r_n}y_n, y) + \dfrac{1}{r_n}\langle y - T_{r_n}y_n, T_{r_n}y_n - Jy_n\rangle \geq 0 \quad \forall y \in C, \\ W_n = \{z \in C\colon \langle x_n - z, J(x_0) - J(x_n)\rangle \geq 0\}, \\ H_n = \{z \in C\colon \phi(z, T_{r_n}y_n) \leq \alpha_n\phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}x_0, \quad n \in Z_+, \end{cases}$$

where $J_{r_n} = (J + r_n M)^{-1}J, \{\alpha_n\}, \{\beta_n\} \subset [0,1],$ and $\{r_n\} \subset (0, \infty)$. They proved a strong convergence theorem and a weak convergence theorem for a common element of the set of solutions of the equilibrium (1.1) and the set of zero points of a maximal monotone operator M in a Banach space under suitable conditions.

Motivated and inspired by above works, the purpose of this article is to introduce and investigate several iterative sequences convergence to a fixed point of a firmly nonexpansive-type mapping (see, Section 2) and a common element of the set of fixed points of the firmly nonexpansive-type mapping and the set of solutions of the equilibrium problem (1.1), respectively. By using generalized projection, strong convergence theorems for firmly nonexpansive-type mappings and equilibrium problems in a Banach space are established under some suitable assumptions.

The remaining of this article is organized as follows. In Section 2, we introduce preliminary results. In Section 3, we investigate the strong convergence theorems for a firmly nonexpansive-type mapping. In Section 4, we explore the strong convergence theorem for a common element of the set of fixed points of the firmly nonexpansive mapping and the set of solutions of equilibrium problem (1.1). Finally, we conclude this article in Section 5.

2. Preliminaries

Throughout this article, we denote by Z_+ and R the set of nonnegative integers and real numbers, respectively. Let C be a nonempty closed convex subset of a Banach space E, and let $T\colon E \to C$ and $F(T) = \{z \in C : Tz = z\}$. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{j(x) \in E^* : \langle j(x), x\rangle = \|j(x)\|\|x\| = \|j(x)\|^2 = \|x\|^2\}.$$

Without confusion, one understands that $\|j(x)\|$ is the E^*-norm and $\|x\|$ is the E-norm. Many properties of the normalized duality mapping J can be found (see, e.g. [1,2,8,10,20,22]).

We list the follows properties:

(p1) $J(x)$ is nonempty for each $x \in E$.
(p2) J is a monotone and bounded operator in Banach space.
(p3) J is a strictly monotone operator in strictly convex Banach space.
(p4) J is the identity operator in Hilbert space.
(p5) If E is a reflexive, smooth and strictly convex Banach space and $J^*\colon E^* \to 2^E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*, JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mapping on E and E^*, respectively.

(p6) If E is a strictly convex Banach space, then J is one to one, i.e.

$$x \neq y \Rightarrow J(x) \cap J(y) = \emptyset.$$

(p7) If E is smooth, then J is single-valued.
(p8) E is a uniformly convex Banach space if and only if E^* is uniformly smooth.
(p9) If E is a uniformly convex and uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on bounded sets of E and $J^{-1} = J^*$ is also uniformly norm-to-norm continuous on bounded sets of E^*.

Let $\phi: E \times E \to R$ be defined as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2 \quad \forall x, y \in E.$$

Remark 2.1 [5,15,20] (i) If E is a reflexive, strictly convex and smooth Banach space, then for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$; (ii) If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$.

Notation $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.

We first recall some definitions and lemmas which are needed in the main results of this work.

Assumption 2.1 *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $f: C \times C \to R$ satisfy the following conditions* (C1)–(C4):

(C1) $f(x, x) = 0$ *for all* $x \in C$.
(C2) f *is monotone, i.e.* $f(x, y) + f(y, x) \leq 0$, *for all* $x, y \in C$.
(C3) f *is upper hemicontinuous, i.e. for all* $x, y, z \in C$, *such that*

$$\limsup_{t \to 0^+} f(tz + (1 - t)x, y) \leq f(x, y).$$

(C4) *For all* $x \in C$, $f(x, \cdot)$ *is convex and lower semicontinuous.*

Definition 2.1 [1,2] We say that $\Pi_C: E \to 2^C$ is a *generalized projection operator* if

$$\Pi_C(x) = \{z \in C: \phi(z, x) \leq \phi(y, x) \ \forall y \in C\}.$$

Remark 2.2 [9,15] If E is a strictly convex Banach space, then the generalized operator $\Pi_C(x)$ is nonempty and single valued.

Remark 2.3 If E is a Hilbert space, then the generalized projection operator is equivalent to the following metric projection operator

$$P_C(x) = \{z \in C: \|x - z\|^2 \leq \|y - z\|^2 \ \forall y \in C\}.$$

Definition 2.2 [18,20] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E.

(1) $T: C \to C$ is called a *firmly nonexpansive-type mapping* if, for all $x, y \in C$,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) - \phi(Tx, x) - \phi(Ty, y),$$

or equivalently,

$$\langle Tx - Ty, J(Tx) - J(Ty)\rangle \le \langle Tx - Ty, J(x) - J(y)\rangle.$$

(2) $T: C \to C$ is called *closed*, if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

(3) $T: C \to C$ is called a *relatively quasi-nonexpansive mapping* if, $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 2.4 If E is a Hilbert space, then firmly nonexpansive-type mapping reduces to nonexpansive mapping.

Example 2.1 [11] If C is a nonempty closed convex subset of a strictly convex, smooth and reflexive Banach space E and P_C is the metric projection of E onto C, then $T = I - P_C$ is a firmly nonexpansive-type mapping.

From the Definition 2.2, the following proposition holds:

PROPOSITION 2.1 *Let T: $C \to C$ be firmly nonexpansive-type mapping such that $F(T) \neq \emptyset$. Then the following statements hold*:

(1) $\phi(p, Tx) + \phi(Tx, x) \le \phi(p, x)$ $\forall p \in F(T)$, $x \in C$;
(2) $\phi(p, Tx) \le \phi(p, x)$ $\forall p \in F(T)$, $x \in C$.

From Proposition 2.1, it is easy to see that a firmly nonexpansive-type mapping is relatively quasi-nonexpansive mapping.

LEMMA 2.1 [13] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

LEMMA 2.2 [1,2,13] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let $x \in C$ and $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, J(x) - J(z)\rangle \le 0 \quad \forall x \in C.$$

LEMMA 2.3 [1,2,13] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y) \quad \forall x, y \in E.$$

LEMMA 2.4 [23] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function g: $[0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|) \quad \forall x, y \in B_r,\ t \in [0, 1],$$

where $B_r = \{z \in E: \|z\| \le r\}$.

LEMMA 2.5 [3] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, f: $C \times C \to R$ satisfy Assumption 2.1 and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, J(z) - J(x)\rangle \ge 0 \quad \forall y \in C.$$

LEMMA 2.6 [22] *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f: $C \times C \to R$ satisfy Assumption 2.1. For $r > 0$ and $x \in E$, define a mapping T_r: $E \to C$ by*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, J(z) - J(x) \rangle \geq 0 \ \forall y \in C \right\} \quad \forall x \in E.$$

Then, the following statements hold:

(i) *T_r is single-valued.*
(ii) *T_r is a firmly nonexpansive-type mapping.*
(iii) *$F(T_r) = EP(f)$, and $EP(f)$ is closed and convex.*

LEMMA 2.7 [18] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let T: $C \to C$ be a relatively quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

PROPOSITION 2.2 *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let T: $C \to C$ be a firmly nonexpansive-type mapping. Then $F(T)$ is closed and convex.*

Proof It directly follows from Lemma 2.7 and Proposition 2.1. This completes the proof. ■

3. Strong convergence theorems for firmly nonexpansive-type mapping

In this section, we shall investigate two iterative sequences' strong convergence to the fixed point of firmly nonexpansive-type mapping in a Banach space under some suitable conditions.

THEOREM 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, T: $C \to C$ be a closed and firmly nonexpansive-type mapping. Let $\{x_n\}$ be a sequence defined as follows*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n J(x_n) + (1 - \beta_n) J(Tx_n)), \\ y_n = J^{-1}(\alpha_n J(x_0) + (1 - \alpha_n) J(Tz_n)), \\ C_n = \{z \in C\colon \langle z - x_n, J(x_0) - J(x_n) \rangle \leq 0\}, \\ Q_n = \{z \in C\colon \phi(z, y_n) \leq \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, z_n)\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \quad n \in Z_+, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ satisfy $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \beta_n = 1$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

Proof By Proposition 2.2, one has that $F(T)$ is closed and convex. Hence $\Pi_{F(T)}$ is well-defined. We now show that C_n and Q_n are nonempty closed and convex. It is easy to check that C_n is closed and convex and Q_n is closed. Since, for any $z \in Q_n$,

$$\begin{aligned} \phi(z, y_n) &\leq \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, z_n), \\ &\Leftrightarrow \alpha_n \phi(z, x_0) + (1 - \alpha_n)\phi(z, z_n) - \phi(z, y_n) \geq 0, \\ &\Leftrightarrow -2\langle z, \alpha_n J(x_0) + (1 - \alpha_n) J(z_n) - J(y_n) \rangle \\ &\quad + \alpha_n \|x_0\|^2 + (1 - \alpha_n)\|z_n\|^2 - \|y_n\|^2 \geq 0. \end{aligned}$$

Let us take arbitrary $v_1, v_2 \in Q_n$. Putting $v = tv_1 + (1-t)v_2 \ \forall t \in [0,1]$. Then, for each $i \in \{1,2\}$,

$$-2\langle v_i, \alpha_n J(x_0) + (1-\alpha_n)J(z_n) - J(y_n)\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2 \geq 0.$$

Therefore, one has

$$\begin{aligned}
&-2\langle v, \alpha_n J(x_0) + (1-\alpha_n)J(z_n) - J(y_n)\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2 \\
&\quad = -2t\langle v_1, \alpha_n J(x_0) + (1-\alpha_n)J(z_n) - J(y_n)\rangle + t(\alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&\qquad - \|y_n\|^2) - 2(1-t)\langle v_2, \alpha_n J(x_0) + (1-\alpha_n)J(z_n) - J(y_n)\rangle \\
&\qquad + (1-t)(\alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2) \geq 0,
\end{aligned}$$

that is, $v \in Q_n$. Thus Q_n is convex. Next, let $\omega \in F(T)$. Since

$$\begin{aligned}
\phi(\omega, y_n) &= \|\omega\|^2 - 2\langle\omega, J(y_n)\rangle + \|y_n\|^2 \\
&= \|\omega\|^2 - 2\langle\omega, \alpha_n J(x_0) + (1-\alpha_n)J(Tz_n)\rangle \\
&\quad + \|\alpha_n J(x_0) + (1-\alpha_n)J(Tz_n)\|^2 \\
&\leq \alpha_n\phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, Tz_n) \\
&\leq \alpha_n\phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, z_n),
\end{aligned}$$

i.e., $\omega \in Q_n$. Thus $F(T) \subset Q_n$ for all $n \in Z_+$.

We next show by induction that $F(T) \subset C_n$ for all $n \in Z_+$. In view of $C_0 = C$, we get $F(T) \subset C_0$. Assume that $F(T) \subset C_k$ for some $k \in Z_+$. Since $x_{k+1} = \Pi_{C_k \cap Q_k} x_0$, and from Lemma 2.2, one can conclude

$$\langle z - x_{k+1}, J(x_0) - J(x_{k+1})\rangle \leq 0 \quad \forall z \in F(T) \subset C_k.$$

This yields $\omega \in C_{k+1}$. Therefore $F(T) \subset C_n \ \forall n \in Z_+$. Moreover, $F(T) \subset C_n \cap Q_n$ for all $n \in Z_+$ and $C_n \cap Q_n$ is nonempty closed and convex, which implies that $\{x_n\}$ is well-defined. By Lemmas 2.2 and 2.3, one has $x_n = \Pi_{C_n} x_0$ and so

$$\phi(x_n, x_0) \leq \phi(\omega, x_0) - \phi(\omega, x_n) \leq \phi(\omega, x_0).$$

Thus $\{\phi(x_n, x_0)\}$ is bounded, and $\{x_n\}$ is also bounded. Since $\phi(\omega, Tx_n) \leq \phi(\omega, x_n)$, $\{Tx_n\}$ is bounded. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n \cap Q_n$, we conclude

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0).$$

Then $\{\phi(x_n, x_0)\}$ is nondecreasing. Thus the limit of $\phi(x_n, x_0)$ exists. It follows from Lemma 2.3 that, for all $m \in Z_+ \setminus \{0\}$,

$$\phi(x_{n+m}, x_n) = \phi(x_{n+m}, \Pi_{C_n} x_0) \leq \phi(x_{n+m}, x_0) - \phi(x_n, x_0),$$

which shows that $\lim_{n\to\infty}\phi(x_{n+m}, x_n) = 0$. Particularly, $\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0$. By Lemma 2.1, we have

$$\lim_{n\to\infty}\|x_{n+m} - x_n\| = \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in C. Let $\lim_{n\to\infty} x_n = \bar{x}$. This together with $x_{n+1} = \Pi_{C_n\cap Q_n} x_0 \in Q_n$ yields

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, z_n).$$

Observe that

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J(z_n)\rangle + \|z_n\|^2\\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J(x_n) + (1-\beta_n)J(Tx_n)\rangle\\ &\quad + \|\beta_n J(x_n) + (1-\beta_n)J(Tx_n)\|^2\\ &\le \beta_n \phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, Tx_n).\end{aligned}$$

Then

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) \le \lim_{n\to\infty} \beta_n \phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, Tx_n) = 0,$$

that is, $\lim_{n\to\infty}\phi(x_{n+1}, z_n) = 0$. Consequently, one has

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) \le \lim_{n\to\infty} \alpha_n \phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, z_n) = 0,$$

that is, $\lim_{n\to\infty}\phi(x_{n+1}, y_n) = 0$. By using Lemma 2.1, one has

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0.$$

Since $\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|$, one gets

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded subset of E,

$$\begin{aligned}\lim_{n\to\infty} \|J(x_n) - J(z_n)\| &= \lim_{n\to\infty} \|J(x_{n+1}) - J(y_n)\|\\ &= \lim_{n\to\infty} \|J(x_{n+1}) - J(z_n)\|\\ &= \lim_{n\to\infty} \|J(x_{n+1}) - J(x_n)\| = 0.\end{aligned}$$

Due to

$$\begin{aligned}\|J(x_{n+1}) - J(y_n)\| &= \|J(x_{n+1}) - \alpha_n J(x_n) - (1-\alpha_n)J(Tz_n)\|\\ &\ge (1-\alpha_n)\|J(x_{n+1}) - J(Tz_n)\| - \alpha_n\|J(x_{n+1}) - J(x_n)\|,\end{aligned}$$

one concludes

$$\|J(x_{n+1}) - J(Tz_n)\| \le \frac{1}{1-\alpha_n}(\|J(x_{n+1}) - J(y_n)\| + \alpha_n\|J(x_{n+1}) - J(x_n)\|).$$

Hence, from $\lim_{n\to\infty}\alpha_n = 0$,

$$\lim_{n\to\infty} \|J(x_{n+1}) - J(Tz_n)\| = 0.$$

Since $J^* = J^{-1}$ is uniformly norm-to-norm continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} \|x_{n+1} - Tz_n\| = 0.$$

By the monotonicity of J and from Definition 2.3, one has

$$0 \leq \langle Tz_n - Tx_n, J(Tz_n) - J(Tx_n)\rangle \leq \langle Tz_n - Tx_n, J(z_n) - J(x_n)\rangle$$

and so,

$$\|Tz_n - Tx_n\|(\|J(z_n) - J(x_n)\| - \|J(Tz_n) - J(Tx_n)\|) \geq 0.$$

Therefore

$$\|J(z_n) - J(x_n)\| \geq \|J(Tz_n) - J(Tx_n)\|.$$

From this it immediately follows that $\lim_{n\to\infty} \|J(Tz_n) - J\,(Tx_n)\| = 0$. Since $J^* = J^{-1}$ is uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|Tz_n - Tx_n\| = 0.$$

Note that

$$\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\| + \|Tz_n - Tx_n\|.$$

This yields that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since T is closed, this together with $\lim_{n\to\infty} x_n = \bar{x}$ implies that $T\bar{x} = \bar{x}$. Then $\bar{x} \in F(T)$.

Let $\bar{\omega} = \Pi_{F(T)}x_0$. From both $\bar{\omega} \in F(T) \subset C_n$ and $x_n = \Pi_{C_n}x_0$, it follows that $\phi(x_n, x_0) \leq \phi(\bar{\omega}, x_0)$. By the weakly lower semicontinuity of the norm,

$$\phi(\bar{x}, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(\bar{\omega}, x_0).$$

Taking into account the uniqueness of $\Pi_{F(T)}x_0$, we get $\bar{x} = \bar{\omega}$. Therefore $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$. This completes the proof. ■

If $\beta_n \equiv 1$ for all $n \in Z_+$ in Theorem 3.1, the following result holds:

Corollary 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, T: $C \to C$ be a closed and firmly nonexpansive-type mapping. Let $\{x_n\}$ be a sequence defined as follows*:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J(x_0) + (1 - \alpha_n)J(Tx_n)), \\ C_n = \{z \in C\colon \langle z - x_n, J(x_0) - J(x_n)\rangle \leq 0\}, \\ Q_n = \{z \in C\colon \phi(z, y_n) \leq \alpha_n\phi(z, x_0) + (1 - \alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n \in Z_+, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies $\lim_{n\to\infty}\alpha_n = 0$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

4. Strong convergence theorem for firmly nonexpansive-type mapping and equilibrium problem

In this section, we shall explore an iterative sequence's strong convergence to a common element of the set of fixed point of firmly nonexpansive-type mapping and the set of solutions for an equilibrium problem (1.1) in a Banach space under some suitable conditions.

THEOREM 4.1 *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f*: $C \times C \to R$ *satisfy Assumption* 2.1 *and let* T: $C \to C$ *be a closed and firmly nonexpansive-type mapping such that* $EP(f) \cap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined as follows*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(Tx_n)), \\ y_n = J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)J(z_n)), \\ u_n \in T_{r_n} y_n = \{z \in C : f(z,y) + \frac{1}{r_n}\langle y - z, J(z) - J(y_n)\rangle \geq 0 \ \forall y \in C\}, \\ C_n = \{z \in C \colon \langle z - x_n, J(x_0) - J(x_n)\rangle \leq 0\}, \\ Q_n = \{z \in C \colon \phi(z,u_n) \leq \alpha_n \phi(z,x_0) + (1-\alpha_n)\phi(z,x_n)\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n \in Z_+, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ *satisfy* $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) \geq a$ *for some* $a > 0$ *and* $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap EP(f)}x_0$.

Proof By Lemma 2.6 and Proposition 2.2, we see that $F(T) \cap EP(f)$ is closed and convex. Hence $\Pi_{F(T)\cap EP(f)}$ is well-defined. We now show that C_n and Q_n are nonempty closed and convex. Clearly, C_n is closed and convex and Q_n is closed. As in the proof of Theorem 3.1, we have Q_n is convex. Therefore, $C_n \cap Q_n$ is closed and convex. Next, let us show that $F(T) \cap EP(f) \subset C_n \cap Q_n \ \forall n \in Z_+$. Let $\omega \in F(T) \cap EP(f)$. Since

$$\begin{aligned} \phi(\omega, z_n) &= \phi(\omega, J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(Tx_n))) \\ &\leq \beta_n \phi(\omega, x_n) + (1-\beta_n)\phi(\omega, Tx_n) \leq \phi(\omega, x_n), \end{aligned}$$

and

$$\begin{aligned} \phi(\omega, y_n) &= \phi(\omega, J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)J(z_n))) \\ &\leq \alpha_n \phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, z_n) \\ &\leq \alpha_n \phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, x_n), \end{aligned}$$

and so, by Lemma 2.7 and Proposition 2.1,

$$\phi(\omega, u_n) = \phi(\omega, T_{r_n} y_n) \leq \phi(\omega, y_n) \leq \alpha_n \phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, x_n).$$

Thus $\omega \in Q_n \ \forall n \in Z_+$. We show by induction that $F(T) \cap EP(f) \subset C_n \ \forall n \in Z_+$. From $C_0 = C$, we get $F(T) \cap EP(f) \subset C_0$. Suppose that $F(T) \cap EP(f) \subset C_k$ for some $k \in Z_+$. From $x_{k+1} = \Pi_{C_k \cap Q_k} x_0$ and Lemma 2.2, it follows that

$$\langle \omega - x_{k+1}, J(x_0) - J(x_{k+1})\rangle \leq 0 \quad \forall \omega \in F(T) \cap EP(f) \subset C_k,$$

which implies that $F(T) \cap EP(f) \subset C_{k+1}$. As a consequence,

$$F(T) \cap EP(f) \subset C_n \quad \forall n \in Z_+.$$

Therefore, $C_n \cap Q_n$ is nonempty closed and convex. This means that $\{x_n\}$ is well-defined, and so

$$F(T) \cap EP(f) \subset C_n \cap Q_n \quad \forall n \in Z_+.$$

By the definition of $C_n, x_n = \Pi_{C_n}x_0$. From Lemma 2.3, it follows that

$$\phi(x_n, x_0) \leq \phi(\omega, x_0) - \phi(\omega, x_n) \leq \phi(\omega, x_0).$$

Hence $\{\phi(x_n, x_0)\}$ is bounded, and $\{Tx_n\}$, $\{x_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{u_n\}$ are also bounded. Let $r = \sup_{n\in Z_+}\{\|x_n\|, \|Tx_n\|\}$. By Lemma 2.4, there is a continuous, strictly increasing and convex function g: $[0, 2r] \to R$ such that $g(0) = 0$ and

$$\begin{aligned}
&\|\beta_n J(x_n) + (1-\beta_n)J(Tx_n)\|^2 \\
&\quad \leq \beta_n\|J(x_n)\|^2 + (1-\beta_n)\|J(Tx_n)\|^2 - \beta_n(1-\beta_n)\,g(\|J(x_n) - J(Tx_n)\|) \\
&\quad = \beta_n\|x_n\|^2 + (1-\beta_n)\|Tx_n\|^2 - \beta_n(1-\beta_n)\,g(\|J(x_n) - J(Tx_n)\|).
\end{aligned}$$

Since $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in C_n \cap Q_n$, we have

$$\phi(x_{n+1}, u_n) \leq \alpha_n\phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, x_n), \tag{4.1}$$

and $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ $\forall n \in Z_+$. Thus, $\{\phi(x_n, x_0)\}$ is nondecreasing. Again from the boundness of $\phi(x_n, x_0)$, it follows that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. Similar to the proof of Theorem 3.1, we have $\lim_{n\to\infty}\phi(x_{n+m}, x_n) = 0$, and $\{x_n\}$ is a Cauchy sequence in C. Let $\lim_{n\to\infty} x_n = \bar{x}$. Then, by $\lim_{n\to\infty}\alpha_n = 0$ and (4.1), we obtain

$$0 \leq \lim_{n\to\infty}\phi(x_{n+1}, u_n) \leq \lim_{n\to\infty}\alpha_n\phi(x_{n+1}, x_0) + \lim_{n\to\infty}(1-\alpha_n)\phi(x_{n+1}, x_n) = 0,$$

i.e., $\lim_{n\to\infty}\phi(x_{n+1}, u_n) = 0$. From Lemma 2.1, it follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_{n+1} - u_n\| = 0.$$

Therefore, from $\|x_n - u_n\| \leq \|x_{n+1} - x_n\| + \|x_n - u_n\|$, one has

$$\lim_{n\to\infty}\|x_n - u_n\| = 0.$$

By the uniformly norm-to-norm continuity of J on bounded subset of E,

$$\lim_{n\to\infty}\|J(x_n) - J(u_n)\| = 0.$$

Since

$$\begin{aligned}
\phi(\omega, z_n) &= \|\omega\|^2 - 2\langle\omega, J(z_n)\rangle + \|z_n\|^2 \\
&= \|\omega\|^2 - 2\langle\omega, \beta_n J(x_n) + (1-\beta_n)J(Tx_n)\rangle + \|\beta_n J(x_n) + (1-\beta_n)J(Tx_n)\|^2 \\
&\leq \|\omega\|^2 - 2\beta_n\langle\omega, J(x_n)\rangle - 2(1-\beta_n)\langle\omega, J(Tx_n)\rangle + \beta_n\|J(x_n)\|^2 \\
&\quad + (1-\beta_n)\|Tx_n\|^2 - \beta_n(1-\beta_n)\,g(\|J(x_n) - J(Tx_n)\|) \\
&\leq \phi(\omega, x_n) - \beta_n(1-\beta_n)\,g(\|J(x_n) - J(Tx_n)\|),
\end{aligned}$$

we obtain

$$\begin{aligned}
\phi(\omega, y_n) &\leq \alpha_n\phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, z_n) \\
&\leq \alpha_n\phi(\omega, x_0) + \phi(\omega, x_n) - \beta_n(1-\beta_n)\,g(\|J(x_n) - J(Tx_n)\|).
\end{aligned}$$

Then, by Lemma 2.6 and Proposition 2.1, we have

$$\begin{aligned}
\phi(\omega, u_n) &= \phi(\omega, T_{r_n}y_n) \leq \phi(\omega, y_n) \\
&\leq \alpha_n\phi(\omega, x_0) + \phi(\omega, x_n) - \beta_n(1-\beta_n)\,g(\|J(x_n) - J(Tx_n)\|)
\end{aligned}$$

and so, $\phi(\omega, y_n) \le \alpha_n\phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, x_n)$. From $\lim\ \inf_{n\to\infty}\beta_n(1-\beta_n) \ge a$, it yields that

$$\begin{aligned} ag(\|J(x_n) - J(Tx_n)\|) &\le \beta_n(1-\beta_n)\, g(\|J(x_n) - J(Tx_n)\|) \\ &\le \alpha_n\phi(\omega, x_0) + \phi(\omega, x_n) - \phi(\omega, u_n) \\ &= \alpha_n\phi(\omega, x_0) + \|x_n\|^2 - \|u_n\|^2 - 2\langle\omega, J(x_n) - J(u_n)\rangle \\ &\le \alpha_n\phi(\omega, x_0) + \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|\omega\|\|J(x_n) - J(u_n)\|. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} g(\|J(x_n) - J(Tx_n)\|) = 0,$$

and consequently,

$$\lim_{n\to\infty} \|J(x_n) - J(Tx_n)\| = 0.$$

Since $J^*{=}J^{-1}$ is uniformly norm-to-norm continuous on bounded subsets of E^*, one has

$$0 = \lim_{n\to\infty} \|J(x_n) - J(Tx_n)\| = \lim_{n\to\infty} \|x_n - Tx_n\|.$$

Since T is closed, this together with $\lim_{n\to\infty} x_n = \bar{x}$ implies that $T\bar{x} = \bar{x}$. Then $\bar{x} \in F(T)$. Let us now show $\bar{x} \in EP(f)$. By Proposition 2.1 and Lemma 2.6, we obtain

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n}y_n, y_n) \\ &\le \phi(\omega, y_n) - \phi(\omega, T_{r_n}y_n) \\ &\le \alpha_n\phi(\omega, x_0) + (1-\alpha_n)\phi(\omega, x_n) - \phi(\omega, u_n) \\ &= \alpha_n(\phi(\omega, x_0) - \phi(\omega, x_n)) + \phi(\omega, x_n) - \phi(\omega, u_n). \end{aligned}$$

Moreover, we can derive that

$$\begin{aligned} \lim_{n\to\infty} \phi(u_n, y_n) &\le \lim_{n\to\infty} [\alpha_n(\phi(\omega, x_0) - \phi(\omega, x_n)) + \phi(\omega, x_n) - \phi(\omega, u_n)] \\ &= \lim_{n\to\infty} \phi(\omega, x_n) - \phi(\omega, u_n) \\ &\le \lim_{n\to\infty} [\|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|\omega\|\|J(x_n) - J(u_n)\|] = 0, \end{aligned}$$

i.e. $\lim_{n\to\infty}\phi(u_n, y_n) = 0$. It follows that $\lim_{n\to\infty}\|u_n - y_n\| = 0$ and $\lim_{n\to\infty}\|J(u_n) - J(y_n)\| = 0$. From both $x_n \to \bar{x}$ and $\lim_{n\to\infty}\|x_n - u_n\| = 0$, it follows that $u_n \to \bar{x}$ and $y_n \to \bar{x}$. In view of $\liminf_{n\to\infty} r_n > 0$, one has

$$\lim_{n\to\infty} \frac{\|J(u_n) - J(y_n)\|}{r_n} = 0.$$

Taking into account the monotonicity of f and

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J(u_n) - J(y_n)\rangle \ge 0 \quad \forall y \in C,$$

one has

$$\frac{1}{r_n}\langle y - u_n, J(u_n) - J(y_n)\rangle \ge -f(u_n, y) \ge f(y, u_n) \quad \forall y \in C.$$

Since f satisfies Assumption 2.1,

$$\begin{aligned} f(y,\bar{x}) &\le \lim_{n\to\infty} f(y,u_n) \\ &\le \frac{1}{r_n}\langle y-u_n, J(u_n)-J(y_n)\rangle \\ &\le \lim_{k\to\infty} \|y-u_n\|\cdot\frac{\|J(u_n)-J(y_n)\|}{r_n} = 0, \end{aligned}$$

i.e. $f(y,\bar{x}) \le 0$ for all $y\in C$. Picking $y\in C$ arbitrarily. Since $\bar{x}\in C$, $ty+(1-t)\bar{x}\in C$ for all $t\in(0,1]$. Therefore $f(ty+(1-t)\bar{x},\bar{x}) \le 0$, and

$$\begin{aligned} 0 &= f(ty+(1-t)\bar{x}, ty+(1-t)\bar{x}) \\ &\le tf(ty+(1-t)\bar{x}, y) + (1-t)f(ty+(1-t)\bar{x},\bar{x}) \\ &\le tf(ty+(1-t)\bar{x}, y). \end{aligned}$$

Moreover, one has

$$f(ty+(1-t)\bar{x}, y) \ge 0.$$

Thus, from Assumption 2.1,

$$0 \le \limsup_{t\to 0+} f(ty+(1-t)\bar{x}, y) \le f(\bar{x}, y),$$

i.e. $f(\bar{x},y)\ge 0$ for all $y\in C$. This means $\bar{x}\in EP(f)$, and so, $\bar{x}\in F(T)\cap EP(f)$.

Let $\bar{\omega} = \Pi_{F(T)\cap EP(f)}x_0$. From both $\bar{\omega}\in F(T)\cap EP(f)\subset C_n$ and $x_n = \Pi_{C_n}x_0$, it follows that $\phi(x_n,x_0)\le\phi(\bar{\omega},x_0)$. By the weakly lower semicontinuity of the norm,

$$\phi(\bar{x},x_0) \le \liminf_{n\to\infty}\phi(x_n,x_0) \le \limsup_{n\to\infty}\phi(x_n,x_0) \le \phi(\bar{\omega},x_0).$$

By the uniqueness of $\Pi_{F(T)\cap EP(f)}\,x_0$, one concludes $\bar{x}=\bar{\omega}$. Therefore $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)}\,x_0$. This completes the proof. ∎

If $\alpha_n\equiv 0$ in Theorem 4.1, we can get the similar result to Theorem 3.1 of [22]:

THEOREM 4.2 *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f: $C\times C\to R$ satisfy Assumption* 2.1 *and let T: $C\to C$ be a closed and firmly nonexpansive-type mapping such that $EP(f)\cap F(T)\neq\emptyset$. Let $\{x_n\}$ be a sequence defined as follows*:

$$\begin{cases} x_0\in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(Tx_n)), \\ u_n\in T_{r_n}y_n = \left\{z\in C\colon f(z,y) + \dfrac{1}{r_n}\langle y-z, J(z)-J(y_n)\rangle \ge 0\ \forall y\in C\right\}, \\ C_n = \{z\in C\colon \langle z-x_n, J(x_0)-J(x_n)\rangle\le 0\}, \\ Q_n = \{z\in C\colon \phi(z,u_n)\le\phi(z,x_n)\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n}x_0, \quad n\in Z_+, \end{cases}$$

where $\{\beta_n\}\subset[0,1]$ satisfies $\liminf_{n\to\infty}\beta_n(1-\beta_n)\ge a$ for some $a>0$ and $\{r_n\}\subset(0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)}\,x_0$.

If E is a Hilbert space, $f(y_n, y) \geq 0$ for all $y \in C$, $n \in Z_+$ in Theorem 4.2, *then we get the modified result* [17]:

COROLLARY 4.1 [17] *Let C be a nonempty closed convex subset of a Hilbert space E and T: $C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows*:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ Q_n = \{z \in C \colon \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ C_n = \{z \in C \colon \| y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \in Z_+, \end{cases}$$

where $\{\beta_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \beta_n \ (1-\beta_n) \geq a$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is th metric projection of E onto $F(T)$.

5. Concluding remarks

This article introduces several iterative sequences $\{x_n\}$, and prove the strong convergence of the iterative sequences to a fixed point of firmly nonexpansive-type mapping and a common element of the set of fixed points of firmly nonexpansive-type mapping and the set of solutions of equilibrium problem, respectively. These obtained results extend and modify corresponding results of Nakajo and Takahashi [17] and Takahashi and Zembayashi [22], and the conditions of our results are different from that of corresponding results of Ceng et al. [4]. As a further research, by applying the obtained results, one can study the problem of finding a minimizer of a convex function on E and a solution of variational inequality (inclusion) problems.

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