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EXISTENCE OF SOLUTIONS AND α-WELL-POSEDNESS FOR A SYSTEM OF CONSTRAINED SET-VALUED VARIATIONAL INEQUALITIES

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Abstract. The notions of α-well-posedness and generalized α-well-posedness for a system of constrained variational inequalities involving set-valued mappings (for short, (SCVI)) are introduced in Hilbert spaces. Existence theorems of solutions for (SCVI) are established by using penalty techniques. Metric characterizations of α-well-posedness and generalized α-well-posedness, in terms of the approximate solutions sets, are presented. Finally, the equivalences between (generalized) α-well-posedness for (SCVI) and existence and uniqueness of its solutions are also derived under quite mild assumptions.

1. **Introduction.** Variational inequalities which introduced by Stampacchia [28] in 1964, are among the most interesting and intensively studied classes of mathematics problems and have wide applications in the fields of optimization and control, economics, electrical networks, game theory, engineering science and transportation equilibria etc. For the past decades, many existence results and iterative algorithms for variational inequality, equilibrium and variational inclusion problems have been studied (see, e.g., [4, 6, 14, 31, 32] and the references therein). Recently, some new and interesting problems, which are called to be system of variational inequality problems, were introduced and investigated. The motivations originate from the fact that under suitable conditions, a Nash equilibrium problem is equivalent to a system of variational inequalities (see, e.g., [13, 14] and the references therein). System of variational inequality problems have been well studied and developed in various aspects. For details, readers are referred to Chen and Wan[5], Cho, Fang, Huang et al. [6], Kim and Kim [14], Mainge [22], Noor and Noor [24] and the

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references therein. In [23], Moudafi and Noor applied the penalty method to prove the existence of solutions for a class of generalized variational inequalities with variational constraints which contained many hierarchical optimization problems such as variational inequalities with equilibrium constraints, variational inequalities with fixed-point problem constraints and variational inequalities with minimization problem constraints as special cases.

On the other hand, well-posedness plays an important role in the stability analysis and numerical methods for equations and optimization theory and applications. It was first introduced by Tykhonov [30] for a minimization problem (for short, (MP)), which has been known as Tykhonov well-posedness. The Tykhonov well-posedness of (MP) implies the existence and uniqueness of solutions of (MP). In many practical situations, the solutions of (MP) are more than one. In this situation, the notion of Tykhonov well-posedness in the generalized sense was introduced, which implies the existence of solutions of (MP). Since then, many authors investigated the well-posedness for optimization problems (see, e.g., [7, 8, 19] and the references therein). In 1981, Lucchetti and Patrone [21] introduced the first notion of well-posedness for variational inequalities, which is a generalization of the Tykhonov well-posedness of (MP). Lignola and Morgan [17] also introduced another notion of well-posedness for variational inequalities, which is distinct from that in Lucchetti and Patrone [21]. For the past decades, well-posedness and well-posedness in the generalized sense for variational inequalities and equilibrium problems have been studied (see, e.g., [3, 9, 10, 12, 16, 26] and the references therein). Lignola and Morgan [18] introduced α-well-posedness for variational inequalities and Nash equilibrium, further, they also discussed the α-well-posedness for parametric noncooperative games and for optimization problems with constraints defined by parametric Nash equilibria, gave some classes of functions that ensure these types of well-posedness in [19]. Peng and Tang [27] studied the α-well-posedness, $L-\alpha$-well-posedness, α-well-posedness in the generalized sense and $L-\alpha$-well-posedness in the generalized sense for mixed quasi-variational-like inequality problems, and presented some metric characterizations for these well-posedness. In 2010, Hu et al. [11] investigated the well-posedness and generalized well-posedness for a system of equilibrium problems, obtained some metric characterizations for these well-posedness. They also proved that the well-posedness of system of equilibrium problems is equivalent to the existence and uniqueness of its solution. Peng and Wu [26] also explored the generalized Tykhonov well-posedness for system of vector quasi-equilibrium problems, and gave some metric characterizations for these well-posedness in locally convex Hausdorff topological vector spaces.

It is natural to raise a question: Whether the existence of solutions and well-posedness can be applied to a class of system of variational inequalities involving set-valued mappings related to hierarchical optimization problems or not ?

Inspired and motivated by the works mentioned above, the purpose of this paper is to introduce and investigate a system of constrained variational inequalities involving set-valued mappings (for short, (SCVI)) in Hilbert spaces. Firstly, existence theorems of solutions for (SCVI) are established by using penalty techniques. Moreover, we also introduce the notions of α-well-posedness and generalized α-well-posedness for (SCVI) in Hilbert spaces. Some metric characterizations of α-well-posedness and generalized α-well-posedness, in terms of the approximate solutions sets, are presented. Finally, the equivalences between (generalized) α-well-posedness for (SCVI) and existence and uniqueness of its solutions are also derived under quite mild assumptions.

This paper is organized as follows. In Section 2, we introduce the problem (SCVI), recall some basic definitions and lemmas. In Section 3, by employing the penalty method, we investigate the existence of solutions for (SCVI) under some suitable conditions. In Section 4, we present and study the definition of α-well-posedness for (SCVI) by using an associated auxiliary problem, and then discuss some characterizations of the α-well-posedness and generalized α-well-posedness for (SCVI).

2. **Preliminaries.** Throughout this paper, without other specifications, let R be the set of real numbers, E be a Hilbert space, denote its scalar product by $\langle \cdot, \cdot \rangle$ and its norm by $\|\cdot\|$. For any $(x, y), (x', y') \in E \times E$, we define that $\|(x, y) - (x', y')\| = \|x - x'\| + \|y - y'\|$. Let $T_1, T_2 : E \times E \to 2^E$ and $M : E \to 2^E$ be set-valued mappings, where 2^E stands for the family of all nonempty subsets of E, denote the graph of M by $G(M) = \{(x, \mu) \in E \times E : \mu \in M(x)\}$. Recall that a mapping $T : E \to E$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. Let $\upsilon_1, \upsilon_2 \in E$ and M be a maximal monotone operator. Set $K = \{\nu \in E : 0 \in M(\nu)\}$. We consider the following *system of constrained variational inequalities problem involving set-valued mappings* (for short, (SCVI)) is to find $(x^*, y^*) \in K \times K$ such that there exist $t_1(y^*, x^*) \in T_1(y^*, x^*), t_2(x^*, y^*) \in T_2(x^*, y^*)$,

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^* \rangle \geq 0, \quad \forall x \in K, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^* \rangle \geq 0, \quad \forall y \in K, \end{cases} \tag{1}$$

where ρ_1, ρ_2 are two positive constants.

We denote the set of solutions to (SCVI) by S.

Observe that (SCVI) can be equivalently posed as the following variational inclusion problem of finding $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} 0 \in \rho_1 T_1(y^*, x^*) + N_K(x^*) - \upsilon_1, \\ 0 \in \rho_2 T_2(x^*, y^*) + N_K(y^*) - \upsilon_2, \end{cases}$$

where $N_K(x)$ is the normal cone to the set K at $x \in K$, i.e.,

$$N_K(x) =: \{\xi \in E : \langle \xi, y - x \rangle \leq 0, \forall y \in K\}.$$

Special cases:

(I) If the set-valued mappings T_1 and T_2 are the same as the set-valued mapping $T : E \to 2^E$, $\rho_1 = \rho_2 = 1$ and $\upsilon_1 = \upsilon_2 = \upsilon$, then the problem (SCVI) is equivalent to find $x^* \in M^{-1}(0)$ such that there exists $t(x^*) \in T(x^*)$,

$$\langle t(x^*) - \upsilon, x - x^* \rangle \geq 0, \quad \forall x \in M^{-1}(0), \tag{2}$$

which is so-called *variational inequalities with variational problem constraints* (for short, (VIVPC)). (VIVPC) contain many hierarchical optimization problems (see, e.g., [23]).

(i) If $M := M_g$, where $g : E \times E \to R$ is a monotone function, M_g is the associated m-monotone operator, that is, $\omega \in M_g(x) \Leftrightarrow g(x, y) + \langle \omega, x - y \rangle \geq 0$, for all $\omega \in E$, then (VIVPC) is equivalent to find $x^* \in EP$ such that there exists $t(x^*) \in T(x^*)$,

$$\langle t(x^*) - \upsilon, x - x^* \rangle \geq 0, \quad \forall x \in EP, \tag{3}$$

where $EP = M_g^{-1}(0) = \{\bar{x} \in E : g(\bar{x}, y) \geq 0, \quad \forall y \in E\}$, which is called *variational inequalities with equilibrium constraints.*

(ii) If $M := I - F$, where $F : E \to E$ is a nonexpansive mapping, it is well known that M is m-monotone, then (VIVPC) is equivalent to find $x^* \in \mathrm{Fix}(F)$ such that there exists $t(x^*) \in T(x^*)$,

$$\langle t(x^*) - \upsilon, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(F), \tag{4}$$

where $\mathrm{Fix}(F) = M^{-1}(0) = \{\bar{x} \in E : F(\bar{x}) = \bar{x}\}$, which is called *variational inequalities with fixed point problem constraints.*

(iii) If $\phi : E \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function, it is well known that $M := \partial\phi$ is m-monotone, where $\partial\phi$ is the subdifferential of ϕ, then (VIVPC) is equivalent to find $x^* \in \mathrm{Argmin}(\phi)$ such that there exists $t(x^*) \in T(x^*)$,

$$\langle t(x^*) - \upsilon, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Argmin}(\phi), \tag{5}$$

which is called *variational inequalities with minimization problem constraints.*

(II) If T_1, T_2 are two single-valued mapping, and $T_1(y, x) = T_1'(x, y)$, then the problem (SCVI) is equivalent to find $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \rho_1 T_1'(x^*, y^*) - \upsilon_1, x - x^* \rangle \geq 0, & \forall x \in K, \\ \langle \rho_2 T_2(x^*, y^*) - \upsilon_2, y - y^* \rangle \geq 0, & \forall y \in K, \end{cases} \tag{6}$$

where ρ_1, ρ_2 are two positive constants, which is studied by Moudafi and Noor [23] and Tang and Liu [29].

We first recall some definitions and lemmas which are needed in the main results of this work.

Definition 2.1. Let $M : E \to 2^E$ be a set-valued mapping. M is said to be

(i) *monotone* if, for any $x, y \in E, u \in M(x)$ and $v \in M(y)$,

$$\langle u - v, x - y \rangle \geq 0.$$

(ii)*m-monotone(maximal-monotone)* if M is monotone and its graph $G(M) = \{(x, \zeta) : \zeta \in M(x)\}$ is not properly contained in the graph of any other monotone.

Remark 1. (i) If M is m-monotone, then the set $M^{-1}(0) = \{x \in E : 0 \in M(x)\}$ is nonempty closed and convex;

(ii) M is m-monotone if and only if M is monotone and $(I + \sigma M)(E) = E$ holds for every $\sigma > 0$, where I is the identity operator on E.

Definition 2.2. [14] Let K_1, K_2 be nonempty subsets of E. The *Hausdorff pesudo-metric* $H(\cdot, \cdot)$ between K_1 and K_2 is defined by

$$H(K_1, K_2) = \max\{e(K_1, K_2), e(K_2, K_1)\},$$

where $e(K_1, K_2) = \sup_{k_1 \in K_1} d(k_1, K_2)$ with $d(k_1, K_2) = \inf_{k_2 \in K_2} \|k_1 - k_2\|$.

It is worth mentioning that if the domain of the Hausdorff pesudo-metric $H(\cdot, \cdot)$ is confined to closed bounded subsets of E, then $H(\cdot, \cdot)$ is the Hausdorff metric.

Now we give some characterizations on the Hausdorff pesudo-metric $H(\cdot, \cdot)$.

Lemma 2.3. *Let K_1, K_2 and K_3 be nonempty subsets of E, $T : E \to E$ be a nonexpansive mapping. Then the following hold:*

(i) $H(K_1, K_1) = 0$;

(ii) $H(K_1, K_3) = H(K_3, K_1)$;

(iii) $H(TK_1, TK_2) \leq H(K_1, K_2)$;

(iv) $H(K_1, K_3) \leq H(K_1, K_2) + H(K_2, K_3)$;

(v) $H(rK_1, rK_3) = |r| H(K_1, K_3), \quad \forall r \in R.$

Proof. (i) and (ii) directly follow from Definition 2.2. We only need to prove (iii), (iv) and (v).

(iii) Since $T : E \to E$ is a nonexpansive mapping, one has

$$e(TK_1, TK_2) = \sup_{k_1 \in K_1} \inf_{k_2 \in K_2} \|Tk_1, Tk_2\| \tag{7a}$$

$$\leq \sup_{k_1 \in K_1} \inf_{k_2 \in K_2} \|k_1 - k_2\| = e(K_1, K_2). \tag{7b}$$

Similarly, we obtain

$$e(TK_2, TK_1) \leq e(K_2, K_1). \tag{8}$$

By Definition 2.2, and from (7) and (8), we have

$$\begin{aligned} H(TK_1, TK_2) &= \max\{e(TK_1, TK_2), e(TK_2, TK_1)\} \\ &\leq \max\{e(K_1, K_2), e(K_2, K_1)\} = H(K_1, K_2), \end{aligned}$$

that is, $H(TK_1, TK_2) \leq H(K_1, K_2)$

(iv) Note that

$$\begin{aligned} e(K_1, K_3) &= \sup_{k_1 \in K_1} \inf_{k_3 \in K_3} \|k_1 - k_3\| \\ &\leq \sup_{k_1 \in K_1} \inf_{k_3 \in K_3} \inf_{k_2 \in K_2} (\|k_1 - k_2\| + \|k_2 - k_3\|) \\ &= \sup_{k_1 \in K_1} \inf_{k_2 \in K_2} \|k_1 - k_2\| + \sup_{k_2 \in K_2} \inf_{k_3 \in K_3} \|k_2 - k_3\| \\ &= e(K_1, K_2) + e(K_2, K_3). \end{aligned}$$

Similarly, we can get

$$e(K_3, K_1) \leq e(K_3, K_2) + e(K_2, K_1).$$

Again from Definition 2.2, we conclude that

$$H(K_1, K_3) \leq H(K_1, K_2) + H(K_2, K_3).$$

(v) For any $r \in R$, one has

$$e(rK_1, rK_3) = |r| \sup_{k_1 \in K_1} \inf_{k_3 \in K_3} \|k_1 - k_3\| = |r| e(K_1, K_3).$$

This follows that $H(rK_1, rK_3) = |r| H(K_1, K_3)$. □

Definition 2.4. Let $\Gamma : E \times E \to 2^E$ be a set-valued mapping. Γ is called:

(i) (ι, γ)-*Lipschitzian* if there exist $\iota > 0$ and $\gamma > 0$ such that

$$H(\Gamma(x_1, y_1), \Gamma(x_2, y_2)) \leq \iota \|x_1 - x_2\| + \gamma \|y_1 - y_2\|, \quad \forall (x_1, y_1), (x_2, y_2) \in E \times E;$$

(ii) β-*strongly monotone with respect to the second argument* if, there exists $\beta > 0$ such that, for each $x \in E$,

$$\langle \eta(x, y_1) - \eta(x, y_2), y_1 - y_2 \rangle \geq \beta \|y_1 - y_2\|^2, \quad \forall y_i \in E, \eta(x, y_i) \in \Gamma(x, y_i), i = 1, 2;$$

(iii) *coercive with respect to the second argument* if, there exists a continuous increasing function $C : R^+ \to R^+$ with $\lim_{t \to +\infty} C(t) = +\infty$ such that for each $x \in E$,

$$\langle \eta(x, y), y \rangle \geq C(\|y\|)\|y\|, \quad \forall y \in E, \eta(x, y) \in \Gamma(x, y).$$

Definition 2.5. [15] Let K be a nonempty and bounded subset of E. The *Kuratowski measure of noncompactness* μ of the set K is defined by

$$\mu(K) = \inf\{\epsilon > 0 : K \subset \cup_{i=1}^{n} K_i, \mathrm{diam} K_i < \epsilon, i = 1, 2, \cdots, n\},$$

where diam stands for the diameter of a set.

Definition 2.6. [5, 6] Let the set-valued mapping $M : E \to 2^E$ be m-monotone. For any positive number $\delta > 0$, the mapping $R_\delta^M : E \to E$ defined by

$$R_\delta^M(x) = (I + \delta M)^{-1}(x), x \in E,$$

is called the *resolvent operator* associated with M and δ, where I is the identity operator on E.

Remark 2. It is well known that the resolvent operator $R_\delta^M : E \to E$ is single-valued and nonexpansive for all $\delta > 0$, and related to its Yosida approximate, i.e., $M_\delta(x) := \frac{x - R_\delta^M(x)}{\delta}$ with $M_\delta(x) \in M(R_\delta^M(x))$.

Definition 2.7. [1, 23] Let $M_n, M : E \to 2^E$ be m-monotone mappings for $n \in N$. A sequence of set-valued mappings $\{M_n\}$ is said to be *graph-convergent* to the set-valued mapping M if, for any $(x, \nu) \in G(M)$, there exists $(x_n, \nu_n) \in G(M_n)$ such that (x_n, ν_n) converges strongly to (x, ν).

Remark 3. (i) Let $\{t_n\}$ be a sequence of positive real numbers, and M be m-monotone. If $t_n \to +\infty$, then $t_n M$ graph-converges to $N_{M^{-1}(0)}$ (see, e.g., [20]);

(ii) Let $M, M_n, \bar{M}, \bar{M}_n$ be m-monotone mappings for $n \in N$ such that $\bar{M}, \bar{M}_n$ are Lipschitzian with constant $\kappa > 0$ (independent of n), that is, $H(\bar{M}(x), \bar{M}(y)) \le \kappa\|x - y\|$ and $H(\bar{M}_n(x), \bar{M}_n(y)) \le \kappa\|x - y\|$ for all $x, y \in E$. If M_n graph-converges to M and $\bar{M}_n$ graph-converges to $\bar{M}$, then $M_n + \bar{M}_n$ graph-converges to $M + \bar{M}$ (see, e.g., [2]).

3. Existence of solutions for (SCVI).

In this section, we shall investigate the existence of solutions for (SCVI) under some suitable conditions. We firstly consider the following auxiliary variational inclusions problem (for short, (AVI)) related to (SCVI): find $(x_\epsilon, y_\delta) \in E \times E$ such that

$$\begin{cases} \upsilon_1 \in \rho_1 T_1(y_\delta, x_\epsilon) + \frac{1}{\epsilon} P(x_\epsilon), \\ \upsilon_2 \in \rho_2 T_2(x_\epsilon, y_\delta) + \frac{1}{\delta} P(y_\delta), \end{cases} \tag{9}$$

where $\rho_1, \rho_2, \epsilon, \delta$ are positive real numbers, $\upsilon_1, \upsilon_2 \in E$ and $P := M_1 = I - R_1^M$ which will act as a penalty operator of the set of zeroes to M. For example, if $M = N_\Lambda$ the normal cone to a closed convex set Λ, then $P := I - P_\Lambda$ is the classical penalty operator of Λ (see, e.g., [25]), where P_Λ is the metric projection operator from E onto the closed convex set Λ.

It is clear that the *Yosida approximate* $P := M_1 = I - R_1^M$ is m-monotone. Moreover, we can reformulate (AVI) as the following equivalent form:

$$\begin{cases} x_\epsilon \in R_{\epsilon^{-1}}^P(x_\epsilon + \upsilon_1 - \rho_1 T_1(y_\delta, x_\epsilon)), \\ y_\delta \in R_{\delta^{-1}}^P(y_\delta + \upsilon_2 - \rho_2 T_2(x_\epsilon, y_\delta)). \end{cases} \tag{10}$$

Using the equivalent form of (AVI), we suggest the following iterative method for solving (AVI).

Algorithm 1. *Let E be a real Hilbert space, $\rho_1, \rho_2, \epsilon, \delta > 0$, and let $T_1, T_2 : E \times E \to 2^E$ be two set-valued mappings, $M : E \to 2^E$ be m-monotone and its Yosida approximate $P = I - R_1^M$. For any given points $x_0, y_0 \in E$, define sequences $\{x_n\}$ and $\{y_n\}$ in E by the following algorithm*

$$\begin{cases} y_{n+1} \in R^P_{\delta^{-1}}(y_n + \upsilon_2 - \rho_2 T_2(x_n, y_n)), \\ x_{n+1} \in R^P_{\epsilon^{-1}}(x_n + \upsilon_1 - \rho_1 T_1(y_n, x_n)), \quad n = 0, 1, 2, ... \end{cases} \tag{11}$$

Remark 4. Algorithm 1 is an explicit iterative method, which is distinct from the implicit iterative algorithm(3.7) of Moudafi and Noor [23].

Next we show the existence of solutions to (AVI) by Algorithm 1.

Theorem 3.1. *Let E be a real Hilbert space, ρ_1, ρ_2 be two positive constants, $M : E \to 2^E$ be m-monotone, and let $T_i : E \times E \to 2^E$ be (ι_i, γ_i)-Lipschitzian and β_i-strongly monotone with respect to the second argument, $i = 1, 2$ such that* $\max\{\rho_1\iota_1 + \sqrt{1 - 2\rho_2\beta_2 + \rho_2^2\gamma_2^2}, \rho_2\iota_2 + \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2}\} < 1$. *Then for any given $\epsilon > 0, \delta > 0$, there exists (x_ϵ, y_δ) which is a solution of the problem (AVI).*

Proof. For any given $x_0, y_0 \in E$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 1. Then, from Lemma 2.3,

$$\begin{aligned} & \|x_{n+1} - x_n\| \\ \leq\ & H(R^P_{\epsilon^{-1}}(x_n + \upsilon_1 - \rho_1 T_1(y_n, x_n)), R^P_{\epsilon^{-1}}(x_{n-1} + \upsilon_1 - \rho_1 T_1(y_{n-1}, x_{n-1}))) \\ \leq\ & H(x_n - \rho_1 T_1(y_n, x_n), x_{n-1} - \rho_1 T_1(y_{n-1}, x_{n-1})) \\ \leq\ & H(x_n - \rho_1 T_1(y_n, x_n), x_n - \rho_1 T_1(y_{n-1}, x_n)) \\ & + H(x_n - \rho_1 T_1(y_{n-1}, x_n), x_{n-1} - \rho_1 T_1(y_{n-1}, x_{n-1})) \\ =\ & \rho_1 H(T_1(y_n, x_n), T_1(y_{n-1}, x_n)) \\ & + H(x_n - \rho_1 T_1(y_{n-1}, x_n), x_{n-1} - \rho_1 T_1(y_{n-1}, x_{n-1})). \end{aligned}$$

Since $T_i : E \times E \to 2^E$ is (ι_i, γ_i)-Lipschitzian and β_i-strongly monotone with respect to the second argument, $i = 1, 2$, we have

$$\begin{aligned} & e(x_n - \rho_1 T_1(y_{n-1}, x_n), x_{n-1} - \rho_1 T_1(y_{n-1}, x_{n-1})) \\ =\ & \sup_{t_1(y_{n-1}, x_n) \in T_1(y_{n-1}, x_n)} \ \inf_{t_1'(y_{n-1}, x_{n-1}) \in T_1(y_{n-1}, x_{n-1})} \|x_n - x_{n-1} \\ & - \rho_1(t_1(y_{n-1}, x_n) - t_1'(y_{n-1}, x_{n-1}))\| \end{aligned}$$

and

$$\begin{aligned} & \|x_n - x_{n-1} - \rho_1(t_1(y_{n-1}, x_n) - t_1'(y_{n-1}, x_{n-1}))\|^2 \\ =\ & \|x_n - x_{n-1}\|^2 - 2\rho_1\langle t_1(y_{n-1}, x_n) - t_1'(y_{n-1}, x_{n-1}), x_n - x_{n-1}\rangle \\ & + \rho_1^2\|t_1(y_{n-1}, x_n) - t_1'(y_{n-1}, x_{n-1})\|^2 \\ \leq\ & (1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2)\|x_n - x_{n-1}\|^2. \end{aligned}$$

Consequently,

$$\begin{aligned} & e(x_n - \rho_1 T_1(y_{n-1}, x_n), x_{n-1} - \rho_1 T_1(y_{n-1}, x_{n-1})) \\ \leq\ & \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2}\|x_n - x_{n-1}\|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & e(x_{n-1} - \rho_1 T_1(y_{n-1}, x_{n-1}), x_n - \rho_1 T_1(y_{n-1}, x_n)) \\ \leq\ & \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2}\|x_n - x_{n-1}\|. \end{aligned}$$

From Definition 2.2 and the above two inequalities, it follows that

$$\begin{aligned} & H(x_n - \rho_1 T_1(y_{n-1}, x_n), x_{n-1} - \rho_1 T_1(y_{n-1}, x_{n-1})) \\ \leq & \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2}\|x_n - x_{n-1}\|. \end{aligned}$$

Thus, we get

$$\|x_{n+1} - x_n\| \leq \rho_1\iota_1\|y_n - y_{n-1}\| + \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2}\|x_n - x_{n-1}\|. \tag{12}$$

Similarly, one can conclude

$$\|y_{n+1} - y_n\| \leq \rho_2\iota_2\|x_n - x_{n-1}\| + \sqrt{1 - 2\rho_2\beta_2 + \rho_2^2\gamma_2^2}\|y_n - y_{n-1}\|. \tag{13}$$

In view of $\max\{\rho_1\iota_1 + \sqrt{1 - 2\rho_2\beta_2 + \rho_2^2\gamma_2^2}, \rho_2\iota_2 + \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2}\} < 1$. Set $\Upsilon = \max\{\rho_1\iota_1 + \sqrt{1 - 2\rho_2\beta_2 + \rho_2^2\gamma_2^2}, \rho_2\iota_2 + \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2}\}$. Then $0 \leq \Upsilon < 1$. From both (12) and (13), it follows that

$$\begin{aligned} & \|(x_{n+1}, y_{n+1}) - (x_n, y_n)\| \\ = & \|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\ \leq & (\rho_1\iota_1 + \sqrt{1 - 2\rho_2\beta_2 + \rho_2^2\gamma_2^2})\|y_n - y_{n-1}\| \\ & +(\rho_2\iota_2 + \sqrt{1 - 2\rho_1\beta_1 + \rho_1^2\gamma_1^2})\|x_n - x_{n-1}\| \\ \leq & \Upsilon\|(x_n, y_n) - (x_{n-1}, y_{n-1})\|, \end{aligned}$$

which implies that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Therefore, there exists $(x_\epsilon, y_\delta) \in E \times E$ such that $(x_n, y_n) \to (x_\epsilon, y_\delta)$. By Remark 2 and (11), one has

$$\begin{cases} y_\delta \in R^P_{\delta^{-1}}(y_\delta + \upsilon_2 - \rho_2 T_2(x_\epsilon, y_\delta)), \\ x_\epsilon \in R^P_{\epsilon^{-1}}(x_\epsilon + \upsilon_1 - \rho_1 T_1(y_\delta, x_\epsilon)), \end{cases}$$

that is,

$$\begin{cases} \upsilon_1 \in \rho_1 T_1(y_\delta, x_\epsilon) + \frac{1}{\epsilon}P(x_\epsilon), \\ \upsilon_2 \in \rho_2 T_2(x_\epsilon, y_\delta) + \frac{1}{\delta}P(y_\delta). \end{cases}$$

Therefore, (x_ϵ, y_δ) is a solution of (AVI). □

Theorem 3.2. *Let E be a real Hilbert space, $M : E \to 2^E$ be m-monotone with $0 \in M^{-1}(0)$, $T_1, T_2 : E \times E \to 2^E$ be coercive and m-monotone with respect to the second argument. Assume that all the conditions of Theorem 3.1 are satisfied. Then (SCVI) has a solution.*

Proof. By the method of Moudafi and Noor [23]. For any given $\upsilon_1, \upsilon_2 \in E$ and $\epsilon, \delta > 0$, from Theorem 3.1, we know that (AVI) has a solution (x_ϵ, y_δ), that is, there exist $t_1(y_\delta, x_\epsilon) \in T_1(y_\delta, x_\epsilon), t_2(x_\epsilon, y_\delta) \in T_1(x_\epsilon, y_\delta)$ such that

$$\begin{cases} \upsilon_1 = \rho_1 t_1(y_\delta, x_\epsilon) + \frac{1}{\epsilon}P(x_\epsilon), \\ \upsilon_2 = \rho_2 t_2(x_\epsilon, y_\delta) + \frac{1}{\delta}P(y_\delta). \end{cases} \tag{14}$$

Moreover, we have

$$\lim_{\epsilon\to 0}\|P(x_\epsilon)\| = \lim_{\delta\to 0}\|P(y_\delta)\| = 0. \tag{15}$$

Since $T_1, T_2 : E \times E \to 2^E$ are coercive with respect to the second argument, we obtain

$$\begin{aligned}\langle \upsilon_1, x_\epsilon\rangle &= \langle \rho_1 t_1(y_\delta, x_\epsilon), x_\epsilon\rangle + \frac{1}{\epsilon}\langle P(x_\epsilon), x_\epsilon\rangle \\ &\geq \rho_1 C(\|x_\epsilon\|)\|x_\epsilon\| + \frac{1}{\epsilon}\langle P(x_\epsilon) - P(0), x_\epsilon - 0\rangle \\ &\geq \rho_1 C(\|x_\epsilon\|)\|x_\epsilon\|\end{aligned}$$

and

$$\begin{aligned}\langle \upsilon_2, y_\delta\rangle &= \langle \rho_2 t_2(x_\epsilon, y_\delta), y_\delta\rangle + \frac{1}{\delta}\langle P(y_\delta), y_\delta\rangle \\ &\geq \rho_2 C(\|y_\delta\|)\|y_\delta\| + \frac{1}{\delta}\langle P(y_\delta) - P(0), y_\delta - 0\rangle \\ &\geq \rho_2 C(\|y_\delta\|)\|y_\delta\|,\end{aligned}$$

which imply that $\{x_\epsilon\}$ and $\{y_\delta\}$ are bounded. Again from (15), we may pick up $x_n = x_{\epsilon_n}, y_n = y_{\delta_n}$ such that x_n and y_n converge weakly to x^* and y^* as $\epsilon_n \to 0, \delta_n \to 0$, respectively.

Now, we show that $T_1(y_n, \cdot)$ graph-converges to $T_1(y^*, \cdot)$. Taking any $(x, \tau) \in G(T_1(y^*, \cdot))$ and $(\hat{x}_n, \tau_n) \in G(T_1(y_n, \cdot))$ such that $\hat{x}_n \to x$ (we may take $\hat{x}_n = x$, if necessary). Since T_1 is (ι_1, γ_1)-Lipschitzian,

$$\|\tau_n - \tau\| \leq H(T_1(y_n, \hat{x}_n), T_1(y^*, x)) \leq \iota_1\|y_n - y^*\| + \gamma_1\|\hat{x}_n - x\|. \tag{16}$$

It follows from (14) that there exists $t_2(x_n, y_n) \in T_2(x_n, y_n)$ such that

$$\upsilon_2 - \frac{1}{\delta_n}P(y_n) = \rho_2 t_2(x_n, y_n). \tag{17}$$

Since T_2 is β_2-strongly monotone with respect to the second argument, for any $t_2(x_n, y^*) \in T_2(x_n, y^*)$, we have

$$\begin{aligned}\rho_2\beta_2\|y^* - y_n\|^2 &\leq \langle \rho_2 t_2(x_n, y^*) - (\upsilon_2 - \frac{1}{\delta_n}P(y_n)), y^* - y_n\rangle \\ &= \langle \rho_2 t_2(x_n, y^*) - \upsilon_2, y^* - y_n\rangle + \frac{1}{\delta_n}\langle P(y_n), y^* - y_n\rangle.\end{aligned}$$

In view of (15), we obtain $\|y^* - y_n\| \to 0$. Together with (16) it follows that $\|\tau_n - \tau\| \to 0$, that is, τ_n converges strongly to τ. Therefore, $T_1(y_n, \cdot)$ graph-converges to $T_1(y^*, \cdot)$. Similarly, $T_2(x_n, \cdot)$ graph-converges to $T_2(x^*, \cdot)$. Note that

$$\begin{aligned}0 = P(x) = (I - R_1^M)(x) \quad &\Leftrightarrow \quad x = R_1^M(x) = (I + M)^{-1}(x) \\ &\Leftrightarrow \quad x \in M^{-1}(0).\end{aligned}$$

From Remark 3, this yields that $\rho_1 T_1(y_n, \cdot) + \frac{1}{\epsilon_n}P$ and $\rho_2 T_2(x_n, \cdot) + \frac{1}{\delta_n}P$ graph-converge to $\rho_1 T_1(y^*, \cdot) + N_{M^{-1}(0)}$ and $\rho_2 T_2(x^*, \cdot) + N_{M^{-1}(0)}$, respectively. Note that

$$\begin{cases} \upsilon_1 \in \rho_1 T_1(y_n, x_n) + \frac{1}{\epsilon_n}P(x_n), \\ \upsilon_2 \in \rho_2 T_2(x_n, y_n) + \frac{1}{\delta_n}P(y_n). \end{cases} \tag{18}$$

Taking the limit in (18), one can obtain

$$\begin{cases} 0 \in \rho_1 T_1(y^*, x^*) + N_{M^{-1}(0)}(x^*) - \upsilon_1, \\ 0 \in \rho_2 T_2(x^*, y^*) + N_{M^{-1}(0)}(y^*) - \upsilon_2. \end{cases} \tag{19}$$

Therefore, (x^*, y^*) is a solution of (SCVI). □

Remark 5. If T_1, T_2 are two single-valued mapping, and $T_1(y,x) = T_1'(x,y)$, then Theorem 3.1 and Theorem 3.2 are reduced to Proposition 3.1 and Theorem 3.1 in [23], respectively.

4. **Characterizations of α-well-posedness for (SCVI).** In this section, we shall present some characterizations of the α-well-posedness and generalized α-well-posedness for (SCVI) by employing an associated auxiliary problem. Let $M : E \to 2^E$ be m-monotone, $K = \{\nu \in E : 0 \in M(\nu)\}$ and $\alpha > 0$. We firstly introduce the following *system of constrained variational inequalities problem* (for short, (SCVI$_\alpha$)) related to (SCVI): find $(x^*, y^*) \in K \times K$ such that there exist $t_1(y^*, x^*) \in T_1(y^*, x^*), t_2(x^*, y^*) \in T_2(x^*, y^*)$,

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^* \rangle + \frac{\alpha}{2}\|x - x^*\|^2 \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^* \rangle + \frac{\alpha}{2}\|y - y^*\|^2 \geq 0, & \forall y \in K, \end{cases} \tag{20}$$

where ρ_1, ρ_2 are two positive constants.

We denote the set of solutions to (SCVI$_\alpha$) by S_α. We now establish the relations between (SCVI) and (SCVI$_\alpha$).

Lemma 4.1. *Let E be a real Hilbert space, $T_1, T_2 : E \times E \to 2^E$ be two set-valued mappings. Then $(x^*, y^*) \in S$ if and only if $(x^*, y^*) \in S_\alpha$.*

Proof. The necessity holds trivially. For the sufficiency, assume that $(x^*, y^*) \in S_\alpha$. Taking any $x, y \in K, \lambda \in (0,1)$, set $x_\lambda = \lambda x + (1-\lambda)x^*, y_\lambda = \lambda y + (1-\lambda)y^*$. Note that $K = \{\nu \in E : 0 \in M(\nu)\}$ is closed and convex. So, $(x_\lambda, y_\lambda) \in K \times K$. By (20), we have

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x_\lambda - x^* \rangle + \frac{\alpha}{2}\|x_\lambda - x^*\|^2 \geq 0, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y_\lambda - y^* \rangle + \frac{\alpha}{2}\|y_\lambda - y^*\|^2 \geq 0, \end{cases}$$

that is,

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^* \rangle + \frac{\alpha\lambda}{2}\|x - x^*\|^2 \geq 0, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^* \rangle + \frac{\alpha\lambda}{2}\|y - y^*\|^2 \geq 0. \end{cases} \tag{21}$$

Let $\lambda \to 0$ in (21). Then

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^* \rangle \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^* \rangle \geq 0, & \forall y \in K. \end{cases} \tag{22}$$

Therefore, $(x^*, y^*) \in S$. □

Next, we introduce the notions of (generalized) α-well-posedness for (SCVI).

Definition 4.2. (i) A sequence $\{(x_n, y_n)\}$ in $K \times K$ is called α-*approximating sequence* for (SCVI) if, there exist $t_1(y_n, x_n) \in T_1(y_n, x_n), t_2(x_n, y_n) \in T_2(x_n, y_n)$ and a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0, \epsilon_n \to 0$ such that

$$\begin{cases} \langle \rho_1 t_1(y_n, x_n) - \upsilon_1, x - x_n \rangle + \frac{\alpha}{2}\|x - x_n\|^2 + \epsilon_n \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x_n, y_n) - \upsilon_2, y - y_n \rangle + \frac{\alpha}{2}\|y - y_n\|^2 + \epsilon_n \geq 0, & \forall y \in K. \end{cases}$$

(ii) A sequence $\{(x_n, y_n)\}$ in $K \times K$ is called α-*approximating sequence* for (SCVI$_\alpha$) if, there exist $t_1(y_n, x_n) \in T_1(y_n, x_n), t_2(x_n, y_n) \in T_2(x_n, y_n)$ and a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0, \epsilon_n \to 0$ such that

$$\begin{cases} \langle \rho_1 t_1(y_n, x_n) - \upsilon_1, x - x_n \rangle + \alpha\|x - x_n\|^2 + \epsilon_n \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x_n, y_n) - \upsilon_2, y - y_n \rangle + \alpha\|y - y_n\|^2 + \epsilon_n \geq 0, & \forall y \in K. \end{cases}$$

Remark 6. (i) It is easy to see that any α-approximating sequence for (SCVI) is α-approximating sequence for (SCVI$_\alpha$);

(ii)If $\alpha = 0$ in Definition 4.2, then the α-approximating sequence for (SCVI) reduces to the approximating sequence for (SCVI);

(iii) The α-approximating sequence for (SCVI) is also called the approximating sequence for (SCVI$_\alpha$).

Definition 4.3. (i) (SCVI) is said to be *α-well-posed* if it has a unique solution and every α-approximating sequence for (SCVI) converges strongly to the unique solution;

(ii) (SCVI) is said to be *generalized α-well-posed* if the solution set S of (SCVI) is nonempty and every α-approximating sequence for (SCVI) has a subsequence which converges strongly to some point of S.

Remark 7. If $\alpha = 0$ in Definition 4.3, then the α-well-posedness and generalized α-well-posedness reduce to the well-posedness and generalized well-posedness, respectively; The generalized α-well-posedness shows that the solution set S of (SCVI) is nonempty and compact.

Definition 4.4. (i) (SCVI$_\alpha$) is said to be *well-posed* if it has a unique solution and every approximating sequence for (SCVI$_\alpha$) converges strongly to the unique solution;

(ii) (SCVI$_\alpha$) is said to be *generalized well-posed* if the solution set S_α of (SCVI$_\alpha$) is nonempty and every approximating sequence for (SCVI$_\alpha$) has a subsequence which converges strongly to some point of S_α;

(iii) (SCVI$_\alpha$) is said to be *α-well-posed* if it has a unique solution and every α-approximating sequence for (SCVI$_\alpha$) converges strongly to the unique solution;

(iv) (SCVI$_\alpha$) is said to be *generalized α-well-posed* if the solution set S_α of (SCVI$_\alpha$) is nonempty and every α-approximating sequence for (SCVI$_\alpha$) has a subsequence which converges strongly to some point of S_α.

Remark 8. From Definitions 4.2-4.4 and Remarks 6-7, it follows that the well-posedness and generalized well-posedness of (SCVI$_\alpha$) imply the α-well-posedness and generalized α-well-posedness of (SCVI), respectively.

The following lemma shows that the relations between the α-well-posedness and generalized α-well-posedness for (SCVI) and the well-posedness and generalized well-posedness for (SCVI$_\alpha$).

Lemma 4.5. *Let E be a real Hilbert space, $T_i : E \times E \to 2^E$ be (ι_i, γ_i)-Lipschitzian, $i = 1, 2$. Then the following hold:*

(i) (SCVI) is α-well-posed if and only if (SCVI$_\alpha$) is well-posed;

(ii) (SCVI) is generalized α-well-posed if and only if (SCVI$_\alpha$) is generalized well-posed.

Proof. It directly follows from Lemma 4.1, Definitions 4.2-4.4 and Remarks 6-7. □

Next, we show that the relations between the α-well-posedness and generalized α-well-posedness for (SCVI) and that for (SCVI$_\alpha$).

Lemma 4.6. *Let E be a real Hilbert space, $T_i : E \times E \to 2^E$ be (ι_i, γ_i)-Lipschitzian, $i = 1, 2$. Then the following hold:*

(i) if (SCVI$_\alpha$) is α-well-posed, then (SCVI) is also α-well-posed;

(ii) if (SCVI$_\alpha$) is generalized α-well-posed, then (SCVI) is also generalized α-well-posed.

Proof. It immediately follows from Lemma 4.1, Definition 4.4 and Remarks 6-7. □

In order to illustrate some metric characterizations of α-well-posedness and generalized α-well-posedness, for any $\epsilon \geq 0$, we introduce the following approximating set:

$$Q(\epsilon) = \{(\tilde{x}, \tilde{y}) \in K \times K : \exists t_1(\tilde{y}, \tilde{x}) \in T_1(\tilde{y}, \tilde{x}), t_2(\tilde{x}, \tilde{y}) \in T_2(\tilde{x}, \tilde{y}),$$
$$\langle \rho_1 t_1(\tilde{y}, \tilde{x}) - \upsilon_1, x - \tilde{x} \rangle + \frac{\alpha}{2}\|x - \tilde{x}\|^2 + \epsilon \geq 0,$$
$$\langle \rho_2 t_2(\tilde{x}, \tilde{y}) - \upsilon_2, y - \tilde{y} \rangle + \frac{\alpha}{2}\|y - \tilde{y}\|^2 + \epsilon \geq 0, \quad \forall x, y \in K\}.$$

It is easy to see that $Q(\epsilon_1) \subset Q(\epsilon_2)$ for all $\epsilon_1, \epsilon_2 \geq 0$ with $\epsilon_1 \leq \epsilon_2$.

Theorem 4.7. *Let E be a real Hilbert space, and $T_i : E \times E \to 2^E$ be (ι_i, γ_i)-Lipschitzian, $i = 1, 2$. Then (SCVI) is α-well-posed if and only if $Q(\epsilon) \neq \emptyset$ for all $\epsilon > 0$ and diam$(Q(\epsilon)) \to 0$ as $\epsilon \to 0$.*

Proof. If (SCVI) is α-well-posed, then there exists a unique $(x^*, y^*) \in K \times K$ such that there exist $t_1(y^*, x^*) \in T_1(y^*, x^*), t_2(x^*, y^*) \in T_2(x^*, y^*)$,

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^* \rangle \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^* \rangle \geq 0, & \forall y \in K. \end{cases}$$

Moreover,

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^* \rangle + \frac{\alpha}{2}\|x - x^*\|^2 \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^* \rangle + \frac{\alpha}{2}\|y - y^*\|^2 \geq 0, & \forall y \in K. \end{cases}$$

Therefore, $(x^*, y^*) \in Q(\epsilon), \forall \epsilon > 0$, that is, $Q(\epsilon) \neq \emptyset$.

Suppose that diam$(Q(\epsilon)) \not\to 0$ as $\epsilon \to 0$. Then there exists $\sigma > 0$, for any sequence $\{\epsilon_n\}$ with $\epsilon_n > 0, \epsilon_n \to 0$ and $(x_n, y_n), (x_n', y_n') \in Q(\epsilon_n)$ such that

$$\|(x_n', y_n') - (x_n, y_n)\| > \sigma, \quad \forall n \in N. \tag{23}$$

Clearly, $\{(x_n, y_n)\}, \{(x_n', y_n')\}$ are α-approximating sequences. Then (x_n, y_n) and (x_n', y_n') must converge strongly to the unique solution $(x^*, y^*) \in S$, which contradict (23).

For the sufficiency, assume that $Q(\epsilon) \neq \emptyset, \forall \epsilon > 0$ and diam$(Q(\epsilon)) \to 0$ as $\epsilon \to 0$. Let $\{(x_n, y_n)\}$ be an α-approximating sequence for (SCVI). Then there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0, \epsilon_n \to 0$ such that there exist $t_1(y_n, x_n) \in T_1(y_n, x_n), t_2(x_n, y_n) \in T_2(x_n, y_n)$,

$$\begin{cases} \langle \rho_1 t_1(y_n, x_n) - \upsilon_1, x - x_n \rangle + \frac{\alpha}{2}\|x - x_n\|^2 + \epsilon_n \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x_n, y_n) - \upsilon_2, y - y_n \rangle + \frac{\alpha}{2}\|y - y_n\|^2 + \epsilon_n \geq 0, & \forall y \in K, \end{cases} \tag{24}$$

that is, $(x_n, y_n) \in Q(\epsilon_n)$ for $n \in N$. In the light of diam$(Q(\epsilon)) \to 0$ as $\epsilon \to 0$, the solution of (SCVI$_\alpha$) is unique and so, $\{(x_n, y_n)\}$ is a Cauchy sequence which converges strongly to $(x^*, y^*) \in K \times K$. Since T_i is (ι_i, γ_i)-Lipschitzian, $i = 1, 2$, and from (24), there exist $t_1(y^*, x^*) \in T_1(y^*, x^*), t_2(x^*, y^*) \in T_2(x^*, y^*)$,

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^* \rangle + \frac{\alpha}{2}\|x - x^*\|^2 \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^* \rangle + \frac{\alpha}{2}\|y - y^*\|^2 \geq 0, & \forall y \in K. \end{cases} \tag{25}$$

Consequently, (x^*, y^*) is the unique solution of (SCVI$_\alpha$). By Lemma 4.1, $(x^*, y^*) \in S_\alpha$ is also the unique solution of (SCVI). Therefore, (SCVI) is α-well-posed. □

Now we give a Furi-Vignoli type characterization of the generalized α-well-posedness for (SCVI) by using Kuratowski measure of noncompactness instead of the diameter.

Theorem 4.8. *Assume that all the conditions of Theorem 4.7 are satisfied. Then (SCVI) is generalized α-well-posed if and only if $Q(\epsilon) \neq \emptyset, \forall \epsilon > 0$ and $\mu(Q(\epsilon)) \to 0$ as $\epsilon \to 0$.*

Proof. The proof is similar to the proof of Theorem 3.2 in [11] and so it is omitted. □

It is well known that the well-posedness of a optimization problem is equivalent to the existence and uniqueness of its solutions. In [11], Hu, Fang, Huang et al. also obtained the relations among the well-posedness, existence and uniqueness of solution for system of equilibrium problem. Next, we establish analogous results for the α-well-posedness and generalized α-well-posedness for (SCVI).

Theorem 4.9. *Let E be a real Hilbert space, $T_i : E\times E \to 2^E$ be (ι_i, γ_i)-Lipschitzian and β_i-strongly monotone with respect to the second argument, $i = 1, 2$. Assume that the following conditions hold:*

(i) $\alpha - \rho_i\beta_i \leq 0$, $i = 1, 2$;

(ii) for any $x, y, z \in E, t_i(x, y) \in T_i(x, y)$, the mapping $(x, z) \mapsto \langle t_i(x, y), z\rangle$ is convex.

Then (SCVI) is α-well-posed if and only if it has a unique solution.

Proof. If (SCVI) is α-well-posed, from Definition 4.3, we know that (SCVI) has a unique solution.

Conversely, if (SCVI) has a unique solution (x^*, y^*), then, from Lemma 4.1, $S_\alpha = \{(x^*, y^*)\}$. It follows that there exist $t_1(y^*, x^*) \in T_1(y^*, x^*), t_2(x^*, y^*) \in T_2(x^*, y^*)$ such that

$$\begin{cases} \langle \rho_1 t_1(y^*, x^*) - \upsilon_1, x - x^*\rangle + \frac{\alpha}{2}\|x - x^*\|^2 \geq 0, & \forall x \in K, \\ \langle \rho_2 t_2(x^*, y^*) - \upsilon_2, y - y^*\rangle + \frac{\alpha}{2}\|y - y^*\|^2 \geq 0, & \forall y \in K. \end{cases} \tag{26}$$

For any $x, y, z \in E, t_1(y, x) \in T_1(y, x), t_2(x, y) \in T_2(x, y)$, the mappings $f, g : E \times E \times E \to R$ defined by

$$f(x, y, z) = \langle \rho_2 t_2(x, y) - \upsilon_2, z - y\rangle + \frac{\alpha}{2}\|z - y\|^2,$$
$$g(y, x, z) = \langle \rho_1 t_1(y, x) - \upsilon_1, z - x\rangle + \frac{\alpha}{2}\|z - x\|^2.$$

Obviously, $f(x, y, y) = g(y, x, x) = 0$. Since $T_i : E \times E \to 2^E$ is (ι_i, γ_i)-Lipschitzian and β_i-strongly monotone with respect to the second argument, $i = 1, 2$, it follows that for each $(x, y) \in E \times E$, $f(x, \cdot, \cdot)$ and $g(y, \cdot, \cdot)$ are continuous. Again from (i), we have

$$\begin{aligned} f(x, y, z) + f(x, z, y) &= \langle \rho_2(t_2(x, y) - t_2(x, z)), z - y\rangle + \alpha\|z - y\|^2 \\ &\leq (\alpha - \rho_2\beta_2)\|z - y\|^2 \\ &\leq 0 \end{aligned}$$

and so, $g(y, x, z) + g(x, z, y) \leq 0$. Therefore, $f(x, \cdot, \cdot)$ and $g(x, \cdot, \cdot)$ are monotone and continuous. By (ii), $f(\cdot, y, \cdot)$ and $g(\cdot, x, \cdot)$ are convex and continuous. So, f and g satisfy the conditions of Theorem 4.1 in [11]. By the same argument as Theorem 4.1 in [11], we get that (SCVI$_\alpha$) is well-posed. Together with Lemma 4.5 this yields that (SCVI) is α-well-posed. □

Theorem 4.10. *Assume that there exists $\epsilon_0 > 0$ such that the approximating set $Q(\epsilon_0)$ is nonempty bounded, and all the conditions of Theorem 4.9 are satisfied. Then (SCVI) is generalized α-well-posed.*

Proof. By the same argument of Theorem 4.2 in [11], ($SCVI_\alpha$) is generalized well-posed. From Lemma 4.5 it follows that (SCVI) is generalized α-well-posed. □

Theorem 4.11. *Assume that there exists $\epsilon_0 > 0$ such that the approximating set $Q(\epsilon_0)$ is bounded, and all the conditions of Theorem 4.9 are satisfied. Then (SCVI) is generalized α-well-posed if and only if $S \neq \emptyset$.*

Proof. It directly follows from Definition 2.2 and Theorem 4.10. □

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