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Nonsmooth multiobjective optimization problems and weak vector quasi-variational inequalities

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Abstract In this article, we investigate a nonsmooth multiobjective optimization problem (MOP) under generalized invexity. First, the Kuhn–Tucker type optimality conditions for MOP are obtained. Furthermore, the relationships between weakly efficient solutions of MOP and vector valued saddle points of its Lagrange function are established. Last but not the least, the relations between weakly efficient solutions of MOP and solutions of Hartman–Stampacchia weak vector quasi-variational inequalities and Hartman–Stampacchia nonlinear weak vector quasi-variational inequalities are also derived under some suitable assumptions. These results extend and improve some known results in the literature.

Keywords Nonsmooth multiobjective optimization problem · Hartman–Stampacchia (nonlinear) weak vector quasi-variational inequalities · Generalized cone-invex function · Optimality conditions

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1 Introduction

The weak minimum (weakly efficient, weak Pareto) solution is an important concept in mathematical models, economics, decision theory, optimal control, and game theory (see, for example, Chen 1992; Chen et al. 2005, 2011; Chen and Wan 2011; Craven 1995; Wan et al. 2011, etc.). In most works, an assumption of convexity was made for the objective functions. Very recently, some generalized convexity has received more attention (see, for example, Chen et al. 2011; Clarke 1983; Kim 2006; Lee et al. 1996; Luc 1989; Mishra and Wang 2006; Suneja et al. 2008, etc.). A significant generalization of convex functions is invex function introduced first by Hanson (1981), which has greatly been applied in nonlinear optimization and other branches of pure and applied sciences. Craven and Yang (1991) presented generalized cone-invex functions and established a generalized alternative theorem involving nonsmooth functions. Suneja et al. (2008) by introducing the notion of Q-nonsmooth pseudoinvexity, Suneja, Khurana and Vani established some necessary and sufficient optimality conditions for MOP involving Clarke's generalized gradient. Very recently, Chen et al. (2011), studied the optimality conditions for a class of MOP involving cone-invexity, proposed a modified objective function method to solve the MOP, and applied to multiobjective fractional programming problems. Mishra and Wang (2006) established the relationships between the solutions of vector variational-like inequalities and (weakly) efficient solutions of nonsmooth vector optimization problems involving differentiable functions. Wu et al. (2011) also explored the MOP, and the equivalence of weakly efficient solutions, the critical points for the MOP, and solutions for vector variational-like inequalities were established under some suitable conditions. Nonemptiness and compactness of the solutions set for the MOP were proved by using the FKKM lemma (see Appendix) and a fixed point theorem.

Inspired and motivated by aforementioned works, the purpose of this paper was to investigate a nonsmooth multiobjective optimization problems involving generalized cone-invex functions with cone constraints. The Kuhn–Tucker type optimality conditions for MOP are obtained. Furthermore, the relationships between weakly efficient solutions of MOP and vector-valued saddle points of its Lagrange function are established. The relations between weakly efficient solutions of MOP and solutions of Hartman–Stampacchia weak vector quasi-variational inequalities (HSVQI) and Hartman–Stampacchia nonlinear weak vector quasi-variational inequalities (HSNVQI) are also derived under some suitable assumptions, which is distinct from Garzon et al. (2004), Guu and Li (2009), Lee et al. (1996), Li and Li (2008), Mishra and Wang (2006). These results extend and improve corresponding results of Chen et al. (2011), Guu and Li (2009), Lee et al. (1996), Li and Li (2008), Mishra and Wang (2006) to nonsmooth case.

2 Preliminaries

Let R^n be the n-dimensional Euclidean space and $R^n_+ = \{x = (x_1, \ldots, x_n)^{\mathrm{T}} : x_l \geq 0, l = 1, \ldots, n\}$, where the superscript T denotes the transpose. A nonempty subset C of topological vector space X is called a *cone* if $tC \subset C$ for all $t > 0$; C is called a *convex cone* if C is a cone and $C + C \subset C$; C is called a *pointed cone* if C is a cone and $C \bigcap (-C) = \{0\}$.

Throughout this paper, without other specifications, let $Q \subset R^k$ and $S \subset R^m$ be closed convex cones with nonempty interior. Let $\eta : R^n \times R^n \to R^n$ with $\eta(x, x_0) \neq 0$ for some

$x \neq x_0$. Let $f = (f_1, \dots, f_k)^{\mathrm{T}} : R^n \to R^k$, $g = (g_1, \dots, g_m)^{\mathrm{T}} : R^n \to R^m$ and let $e = (1, 1, \dots, 1)^{\mathrm{T}} \in R^k$, where f_i and g_j are locally Lipschitz, $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, m\}$. The dual cone of $K \subset R^n$ is denoted by

$$K^* = \{u \in R^n : x^{\mathrm{T}} u \geq 0,\ \forall x \in K\}.$$

The MOP is defined as follows:

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & -g(x) \in S. \end{aligned}$$

Denote the feasible set of MOP by $F = \{x \in R^n : -g(x) \in S\}$.

We first recall some definitions and lemmas which are needed in our main results.

Definition 2.1 A point $x_0 \in F$ is called a *weakly efficient* (weak minimum) *solution* of MOP if

$$f(x) - f(x_0) \notin -\mathrm{int}Q, \quad \forall x \in F.$$

Denote by F^w the weakly efficient solutions set of MOP.

Definition 2.2 Clarke (1983) A real-valued function $\ell : R^n \to R$ is said to be *locally Lipschitz* if, for any $z \in R^n$, there exist a positive constant κ and a neighborhood N of z such that, for any $x, y \in N$, $|\ell(x) - \ell(y)| \leq \kappa \|x - y\|$, where $\|\cdot\|$ denotes any norm in R^n.

In Clarke (1983), the Clarke's generalized subgradient of ℓ at z is denoted by

$$\partial \ell(z) = \{\xi \in R^n : \ell^\circ(z; d) \geq \xi^{\mathrm{T}} d,\ \forall d \in R^n\},$$

where $\ell^\circ(z; d) = \limsup_{y \to z, t \to 0} \frac{\ell(y+td) - \ell(y)}{t}$.
Clearly, $\ell^\circ(z; d) = \max\{\xi^T d : \xi \in \partial \ell(z)\}$.

Definition 2.3 Let $p : R^n \to R^k$. The *generalized subgradient* of p at $z \in R^n$ is the set

$$\partial p(z) = \{(\zeta_1, \dots, \zeta_k)^{\mathrm{T}} : \zeta_i \in \partial p_i(z),\ i = 1, \dots, k\},$$

where $\partial p_i(z)$ is the Clarke's generalized subgradient of $p_i (i \in \{1, \dots, k\})$ at $z \in R^n$.

Definition 2.4 Craven and Yang (1991) (1) f is said to be *generalized Q-invex* at $z \in R^n$ if there exists $\eta : R^n \times R^n \to R^n$ such that, for all $x \in R^n$ and $\zeta \in \partial f(z)$,

$$f(x) - f(z) - \zeta \eta(x, z) \in Q.$$

(2) f is said to be *generalized Q-pseudoinvex* at $z \in R^n$ if there exists $\eta : R^n \times R^n \to R^n$ such that, for all $x \in R^n$ and $\zeta \in \partial f(z)$,

$$f(x) - f(z) \in -\mathrm{int}Q \Rightarrow \zeta \eta(x, z) \in -\mathrm{int}Q.$$

(3) f is said to be *generalized* Q-(pseudo)invex with respect to η if it is generalized Q-(pseudo)invex at any point $z \in R^n$ with respect to η.

Example 2.1 Let $R^n = R$, $R^k = R^2$ and $Q = -R^2_+$. Assume that $f(x) := (-|x|, 1)^{\mathrm{T}}$ for all $x \in R$. We define the function $\eta : R \times R \to R$ as follows:

$$\eta(x, y) = \begin{cases} |x| - y, & \text{if } \ y > 0, \\ \frac{|x|}{4}, & \text{if } \ y = 0, \\ -|x| - y, & \text{if } \ y < 0. \end{cases}$$

After computation, we know that f is generalized Q-invex at any point $y \in R$ with respect to the η.

Remark 2.5 f is generalized Q-invex and g is generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ if and only if (f, g) is generalized (Q, S)-invex with respect to $\eta : R^n \times R^n \to R^n$.

Remark 2.6 Suneja et al. (2008) defined the following pseudoinvexity: f is said to be *Q-nonsmooth pseudoinvex* at $z \in R^n$ if there exists $\eta : R^n \times R^n \to R^n$ such that, for all $x \in R^n$,

$$f(x) - f(z) \in -\mathrm{int}Q \Rightarrow f^{\circ}(z; \eta(x, z)) \in -\mathrm{int}Q,$$

where $f^{\circ}(z; \eta(x, z)) = (f_1^{\circ}(z; \eta(x, z)), \ldots, f_k^{\circ}(z; \eta(x, z)))^{\mathrm{T}}$ and

$$f_i^{\circ}(z; \eta(x, z)) = \max\{\xi^{\mathrm{T}}\eta(x, z) : \xi \in \partial f_i(z)\}, i = 1, 2, \ldots, k.$$

It is easy to see that, if $Q = R^k_+$, then Q-nonsmooth pseudoinvexity implies generalized Q-pseudoinvexity; if $Q = -R^k_+$, then generalized Q-pseudoinvexity implies Q-nonsmooth pseudoinvexity.

Definition 2.7 MOP is said to satisfy the *linearly independent constraint qualification* at $x \in R^n$ if, $\xi_1, \xi_2, \ldots, \xi_m$ are linearly independent for any $\xi_i \in \partial g_i(x), i \in \{1, \ldots, m\}$.

Definition 2.8 (1) (f, g) is said to be KT-(Q, S)*-pseudoinvex* at y if there exists $\eta : R^n \times R^n \to R^n$ such that, for all $x \in R^n$ and $x \neq y$,

$$f(x) - f(y) \in -\mathrm{int}Q \Rightarrow \begin{cases} \zeta\eta(x, y) \in -\mathrm{int}Q, & \forall\zeta \in \partial f(y), \\ \xi\eta(x, y) \in -S, & \forall\xi \in \partial g(y). \end{cases}$$

(2) (f, g) is said to be KT-(Q, S)*-pseudoinvex* with respect to η if it is KT-(Q, S)-pseudoinvex at each point $z \in R^n$ with respect to η.

Clearly, if f and g are generalized Q-pseudoinvex and generalized S-pseudoinvex with respect to η at y, then (f, g) is KT-(Q, S)*-pseudoinvex* with respect to η at y. If f is generalized Q-invex and g is generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ at y, then (f, g) also is KT-(Q, S)*-pseudoinvex* with respect to η at y.

Example 2.2 Let $R^n = R, R^k = R^m = R^2$ and $Q = S = R^2_+$. Assume that $f(x) := (-|x|, |x|)^{\mathrm{T}}$ and $g(x) := (|x|, -|x|)^{\mathrm{T}}$ for all $x \in R$. By the Definition 2.2 and simple computation, we can obtain that (f, g) is KT-(Q, S)-pseudoinvex with respect to any function $\eta : R \times R \to R$ at any point $y \in R$.

Remark 2.9 If f and g are differentiable on F, then the generalized Q-invexity reduces to the Q-invexity Chen et al. (2011), Li and Li (2008). Moreover, if $Q = R^k_+$ and $S = R^m_+$, then the KT-(Q, S)-pseudoinvexity reduces to the KT-pseudoinvexity (Arana-Jiménez et al. 2008).

Lemma 2.10 Clarke (1983) *If $p_i : R^n \to R(i \in \{1, \ldots, k\})$ are locally Lipschitz, then the following statements are true:*

(1) $\partial(\Sigma_{i=1}^k p_i)(z) \subset \Sigma_{i=1}^k \partial p_i(z)$ *for all* $z \in R^n$*;*
(2) $\partial(\Sigma_{i=1}^k t_i p_i)(z) \subset \Sigma_{i=1}^k t_i \partial p_i(z)$ *for all* $z \in R^n$*, where* $t = (t_1, \ldots, t_k)^{\mathrm{T}} \in R^k$.

Lemma 2.11 Craven (1995) *Let $\Delta \subset R^k$ be a convex cone with $\text{int}\Delta \neq \emptyset$ and Δ^* the dual cone of Δ. Then the following statements are true:*

(1) *If $u \in \text{int}\Delta$, then $x^{\mathrm{T}}u > 0$ for all $x \in \Delta^* \backslash \{0\}$;*
(2) *If $x \in \text{int}\Delta^*$, then $x^{\mathrm{T}}u > 0$ for all $u \in \Delta \backslash \{0\}$.*

Lemma 2.12 Chen et al. (2011), Craven (1995) *Let K be a convex cone of topological vector space X with $\text{int}K \neq \emptyset$. Then, for any $x, y \in X$, the following statements are true:*

(1) *$y - x \in K$ and $y \in -K$ imply $x \in -K$;*
(2) *$y - x \in K$ and $y \in -\text{int}\, K$ imply $x \in -\text{int}\, K$;*
(3) *$y - x \in K$ and $x \notin -\text{int}\, K$ imply $y \notin -\text{int}\, K$.*

3 Optimality conditions for MOP

In this section, we investigate the Kuhn–Tucker type necessary and sufficient optimality conditions for MOP under some suitable conditions.

Theorem 3.1 Suneja et al. (2008) (Kuhn–Tucker type necessary optimality condition) *Suppose that f is generalized Q-invex and g is generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$ and some constraint qualifications are satisfied at $\bar{x} \in F^w$. Then there exist $\lambda \in Q^* \backslash \{0\}$ and $\mu \in S^*$ such that*

$$0 \in \partial f(\bar{x})^{\mathrm{T}}\lambda + \partial g(\bar{x})^{\mathrm{T}}\mu, \quad \mu^{\mathrm{T}} g(\bar{x}) = 0,$$

where $\partial f(\bar{x})^{\mathrm{T}} = \{\zeta^{\mathrm{T}} : \zeta \in \partial f(\bar{x})\}$.

Proof For the proof of this theorem, the reader could refer to Suneja et al. (2008). This completes the proof. □

Theorem 3.2 *(Kuhn–Tucker type sufficient optimality condition) Let (f, g) be KT-(Q, S)-pseudoinvex with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$. Assume that there exist $\lambda \in Q^* \backslash \{0\}$ and $\mu \in S^*$ such that*

$$0 \in \partial f(\bar{x})^{\mathrm{T}}\lambda + \partial g(\bar{x})^{\mathrm{T}}\mu, \quad \mu^{\mathrm{T}} g(\bar{x}) = 0. \tag{1}$$

Then $\bar{x} \in F^w$.

Proof Let $\bar{x} \in F$. Suppose to the contrary that $\bar{x} \notin F^w$. Then there exists $\hat{x} \in F$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int}Q.$$

By the KT-(Q, S)-pseudoinvexity of (f, g) with respect to η at $\bar{x} \in F$, we get

$$\zeta\eta(\hat{x}, \bar{x}) \in -\text{int}Q, \xi\eta(\hat{x}, \bar{x}) \in -S, \quad \forall \zeta \in \partial f(\bar{x}),\ \xi \in \partial g(\bar{x}).$$

Moreover, one has

$$(\lambda^{\mathrm{T}}\zeta + \mu^{\mathrm{T}}\xi)\eta(\hat{x}, \bar{x}) < 0, \quad \forall \zeta \in \partial f(\bar{x}),\ \xi \in \partial g(\bar{x}).$$

which contradicts (1). Therefore, $\bar{x} \in F^w$. This completes the proof. □

Remark 3.3 The KT-(Q, S)-pseudoinvexity with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$ of (f, g) can be replaced by the generalized Q-invexity of f and generalized S-invexity of g with respect to the same η at $\bar{x}$.

Now, we give an example to illustrate the obtained results.

Example 3.1 Let $R^n = R^k = R^m = R^2$ and $Q = S = [0, +\infty) \times (-\infty, 0]$. Let $f : R^n \to R^k$ and $g : R^n \to R^m$. We consider the following problem MOP:

$$\begin{aligned} \min \quad & f(x) := (2x_1, -3x_2^2)^{\mathrm{T}} \\ \text{subject to} \quad & g(x) := (x_1^2 + 4x_1 - 5, -x_2)^{\mathrm{T}} \in -S. \end{aligned}$$

Simple computation allows that the feasible solutions set

$$F = \{(x_1, x_2)^{\mathrm{T}} \in R^2 : -5 \leq x_1 \leq 1, x_2 \leq 0\}$$

and the weakly efficient solutions set $F^w = \{(x_1, x_2)^{\mathrm{T}} : x_1 = -5 \text{ or } x_2 = 0\}$. One can easily verify that f and g are generalized $Q-$invex and generalized $S-$invex with respect to $\eta(x, \bar{x}) := (2^{x_2 \bar{x}_2}(x_1 - \bar{x}_1), x_2 - \pi^{x_1 x_2 - \bar{x}_2})^{\mathrm{T}}$ at $\bar{x}$, respectively, and the linearly independent constraint qualifications hold at $\bar{x}$, where $x = (x_1, x_2)^{\mathrm{T}}$ and $\bar{x} = (-5, 0)^{\mathrm{T}}$. Clearly, there exist $\bar{\lambda} = (3, 0)^{\mathrm{T}} \in Q^*$ and $\bar{\mu} = (1, 0)^{\mathrm{T}} \in S^*$ such that $0 \in \partial f(\bar{x})^{\mathrm{T}} \bar{\lambda} + \partial g(\bar{x})^{\mathrm{T}} \bar{\mu}$ and $\bar{\mu}^{\mathrm{T}} g(\bar{x}) = 0$.

4 Relationship between vector-valued saddle points and weakly efficient solutions for MOP

In this section, let $Q = R^k_+$. We develop the relationship between vector-valued saddle points and weakly efficient solutions for MOP under some suitable assumptions. We now propose the Lagrange function for MOP:

$$L(x, \mu) = (L_1(x, \mu), \ldots, L_k(x, \mu)) = f(x) + \mu^{\mathrm{T}} g(x) e, \quad \forall x \in F,\ \mu \in S^*,$$

where $L_i(x, \mu) = f_i(x) + \mu^{\mathrm{T}} g(x)$, $i \in \{1, 2, \ldots, k\}$ and $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in R^k$.

Definition 4.1 A point $(\bar{x}, \bar{\mu}) \in F \times S^*$ is called a *vector valued saddle point* of the Lagrange function $L(x, \mu)$ if

(1) $L(\bar{x}, \mu) - L(\bar{x}, \bar{\mu}) \notin \mathrm{int} Q$ for all $\mu \in S^*$;
(2) $L(x, \bar{\mu}) - L(\bar{x}, \bar{\mu}) \notin -\mathrm{int} Q$ for all $x \in F$.

Theorem 4.2 *Let $(\bar{x}, \bar{\mu}) \in F \times S^*$ be a vector valued saddle point of $L(x, \mu)$. Then $\bar{x} \in F^w$.*

Proof Let $(\bar{x}, \bar{\mu}) \in F \times S^*$ be a vector-valued saddle point of $L(x, \mu)$. Therefore, from Definition 4.1, there exists $i \in \{1, 2, \ldots, k\}$ such that

$$L_i(\bar{x}, \mu) - L_i(\bar{x}, \bar{\mu}) \leq 0, \quad \forall \mu \in S^*.$$

Moreover, one has

$$\mu^{\mathrm{T}} g(\bar{x}) - \bar{\mu}^{\mathrm{T}} g(\bar{x}) \leq 0, \quad \forall \mu \in S^*,$$

which implies that $\bar{\mu}^{\mathrm{T}} g(\bar{x}) = 0$.

Suppose to the contrary that $\bar{x} \notin F^w$. Then there exists $x_0 \in F$ such that

$$f(x_0) - f(\bar{x}) \in -\mathrm{int} Q. \tag{2}$$

Since $\bar{\mu}^{\mathrm{T}} g(x_0) \leq 0$, it follows from (2) that

$$f(x_0) + \bar{\mu}^{\mathrm{T}} g(x_0) - f(\bar{x}) - \bar{\mu}^{\mathrm{T}} g(\bar{x}) \in -\mathrm{int} Q,$$

which contradicts Definition 4.1. This completes the proof. □

Theorem 4.3 *Suppose that f is generalized Q-invex and g is generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$ and some constraint qualifications are satisfied at $\bar{x}$. Then there exists $\bar{\mu} \in S^*$ such that $(\bar{x}, \bar{\mu})$ is a vector-valued saddle point for $L(x, \mu)$.*

Proof Let $\bar{x} \in F^w$. By Theorem 3.1, there exist $\bar{\lambda} \in Q^* \backslash \{0\}$ and $\bar{\mu} \in S^*$ such that

$$0 \in \partial f(\bar{x})^{\mathrm{T}} \bar{\lambda} + \partial g(\bar{x})^{\mathrm{T}} \bar{\mu}, \quad \bar{\mu}^{\mathrm{T}} g(\bar{x}) = 0. \tag{3}$$

Since $\mu^{\mathrm{T}} g(\bar{x}) \leq 0$ for all $\mu \in S^*$, we have

$$\begin{aligned} L(\bar{x}, \mu) - L(\bar{x}, \bar{\mu}) &= \mu^{\mathrm{T}} g(\bar{x}) e - \bar{\mu}^{\mathrm{T}} g(\bar{x}) e \\ &= \mu^{\mathrm{T}} g(\bar{x}) e \notin \mathrm{int} Q. \end{aligned}$$

Suppose to the contrary that $(\bar{x}, \bar{\mu})$ is not vector-valued saddle point of $L(x, \mu)$. Then there exists $x^0 \in F$ such that

$$\begin{aligned} L(x^0, \bar{\mu}) - L(\bar{x}, \bar{\mu}) &= f(x^0) + \bar{\mu}^{\mathrm{T}} g(x^0) e - f(\bar{x}) - \bar{\mu}^{\mathrm{T}} g(\bar{x}) e \\ &\in -\mathrm{int} Q. \end{aligned}$$

Without loss of generality, let $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_k)^{\mathrm{T}}$ with $\sum_{i=1}^{k} \bar{\lambda}_i = 1$. Then we have

$$\bar{\lambda}^{\mathrm{T}} f(x^0) - \bar{\lambda}^{\mathrm{T}} f(\bar{x}) + \bar{\mu}^{\mathrm{T}} g(x^0) - \bar{\mu}^{\mathrm{T}} g(\bar{x}) < 0. \tag{4}$$

Since f is generalized Q-invex and g is generalized S-invex with respect to the same $\eta : R^n \times R^n \to R^n$ at $\bar{x}$, one has

$$f(x^0) - f(\bar{x}) - \zeta \eta(x^0, \bar{x}) \in Q, \quad \forall \zeta \in \partial f(\bar{x}),$$

and so

$$g(x^0) - g(\bar{x}) - \xi \eta(x^0, \bar{x}) \in S, \quad \forall \xi \in \partial g(\bar{x}).$$

Moreover, we have

$$\bar{\lambda}^{\mathrm{T}} f(x^0) - \bar{\lambda}^{\mathrm{T}} f(\bar{x}) - \bar{\lambda}^{\mathrm{T}} \zeta \eta(x^0, \bar{x}) \geq 0, \quad \forall \zeta \in \partial f(\bar{x}),$$

and so

$$\bar{\mu}^{\mathrm{T}} g(x^0) - \bar{\mu}^{\mathrm{T}} g(\bar{x}) - \bar{\mu}^{\mathrm{T}} \xi \eta(x^0, \bar{x}) \geq 0, \quad \forall \xi \in \partial g(\bar{x}).$$

Therefore, it follows that

$$\begin{aligned} &\bar{\lambda}^{\mathrm{T}} f(x^0) - \bar{\lambda}^{\mathrm{T}} f(\bar{x}) + \bar{\mu}^{\mathrm{T}} g(x^0) - \bar{\mu}^{\mathrm{T}} g(\bar{x}) \\ &\quad \geq (\bar{\lambda}^{\mathrm{T}} \zeta + \bar{\mu}^{\mathrm{T}} \xi) \eta(x^0, \bar{x}), \quad \forall \zeta \in \partial f(\bar{x}),\ \xi \in \partial g(\bar{x}). \end{aligned}$$

Thus it follows from (3) that

$$\bar{\lambda}^{\mathrm{T}} f(x^0) - \bar{\lambda}^{\mathrm{T}} f(\bar{x}) + \bar{\mu}^{\mathrm{T}} g(x^0) - \bar{\mu}^{\mathrm{T}} g(\bar{x}) \geq 0,$$

which contradicts (4). This completes the proof. □

Now, we apply Example 3.1 to illustrate Theorem 4.3.

Example 4.1 We consider the MOP in Example 3.1. It is easy to see that the Lagrange function

$$L(x,\mu) = (\mu_1 x_1^2 + (4\mu_1 + 2)x_1 - \mu_2 x_2 - 5\mu_1, \mu_1 x_1^2 - 3x_2^2 + 4\mu_1 x_1 - \mu_2 x_2 - 5\mu_1)^{\mathrm{T}},$$

where $x \in F = \{(x_1, x_2)^{\mathrm{T}} \in R^2 : -5 \le x_1 \le 1, x_2 \le 0\}$ and $\mu \in S^*$. From Example 3.1, the linearly independent constraint qualifications hold at $\bar{x} = (-5, 0)^{\mathrm{T}}$. One can easily check that there exists $\bar{\mu} = (1, 0)^{\mathrm{T}}$ such that $(\bar{x}, \bar{\mu})$ is a vector-valued saddle point for $L(x,\mu)$.

5 Weak vector quasi-variational inequalities

In this section, we consider the relationships among weakly efficient solutions of MOP, the solutions of a class of HSVQI and that of HSNVQI under some suitable assumptions.

Let F be nonempty set. The HSVQI and HSNVQI are defined as follows:

HSVQI: find $\bar{x} \in F$ such that there exists $\bar{\zeta} \in \partial f(\bar{x})$ and

$$\bar{\zeta}\eta(x,\bar{x}) \notin -\mathrm{int}Q, \quad \forall x \in F.$$

HSNVQI: find $\bar{x} \in F$ such that there exists $\bar{\zeta} \in \partial f(\bar{x})$ and

$$\bar{\zeta}\eta(x,\bar{x}) + f(x) - f(\bar{x}) \notin -\mathrm{int}Q, \quad \forall x \in F.$$

Denote the solutions sets of HSVQI and HSNVQI by S_{QI} and S_{NQI}, respectively.

Theorem 5.1 *Let f be generalized Q-invex with respect to $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F$. Then $\bar{x} \in S_{QI}$ or $\bar{x} \in S_{NQI}$ implies that $\bar{x} \in F^w$.*

Proof If $\bar{x} \in S_{QI}$. Suppose to the contrary that $\bar{x} \notin F^w$. Then there exists $x^0 \in F$ such that

$$f(x^0) - f(\bar{x}) \in -\mathrm{int}Q. \tag{5}$$

By the generalized Q-invexity of f with respect to η at $\bar{x}$, one has

$$f(x^0) - f(\bar{x}) - \zeta\eta(x^0,\bar{x}) \in Q, \quad \forall \zeta \in \partial f(\bar{x}).$$

It follows from (5) that

$$\zeta\eta(x^0,\bar{x}) \in -\mathrm{int}Q \forall \zeta \in \partial f(\bar{x}),$$

which contradicts $\bar{x} \in S_{QI}$.

If $\bar{x} \in S_{NQI}$, then there exists $\bar{\zeta} \in \partial f(\bar{x})$ and

$$\bar{\zeta}\eta(x,\bar{x}) + f(x) - f(\bar{x}) \notin -\mathrm{int}Q, \quad \forall x \in F. \tag{6}$$

Since f is generalized Q-invex with respect to η at $\bar{x}$, we get

$$f(x) - f(\bar{x}) - \zeta\eta(x,\bar{x}) \in Q, \quad \forall \zeta \in \partial f(\bar{x}).$$

Moreover, one has

$$2(f(x) - f(\bar{x})) - (f(x) - f(\bar{x}) + \zeta\eta(x,\bar{x})) \in Q, \quad \forall \zeta \in \partial f(\bar{x}).$$

Thus it follows from (6) that

$$f(x) - f(\bar{x}) \notin -\mathrm{int}Q, \quad \forall x \in F,$$

that is, $\bar{x} \in F^w$. This completes the proof. □

In the following, we show Theorem 5.1 by Example 3.1.

Example 5.1 We consider the MOP in Example 3.1. It follows from Example 3.1 that the feasible solutions set $F = \{(x_1, x_2)^{\mathrm{T}} \in R^2 : -5 \leq x_1 \leq 1, x_2 \leq 0\}$ and f is generalized $Q-$invex with respect to $\eta(x, \bar{x})$, where $\eta(x, \bar{x}) := (2^{x_2\bar{x}_2}(x_1 - \bar{x}_1), x_2 - \pi^{x_1x_2 - \bar{x}_2})^{\mathrm{T}}$ at $\bar{x} = (-5, 0)^{\mathrm{T}}$. Clearly, there exists $\bar{\zeta} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \in \partial f(\bar{x})$ such that

$$\bar{\zeta}\eta(x, \bar{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2^{x_2\bar{x}_2}(x_1 - \bar{x}_1) \\ x_2 - \pi^{x_1x_2 - \bar{x}_2} \end{bmatrix} = \begin{bmatrix} 10 + 2x_1 \\ 0 \end{bmatrix} \notin -\mathrm{int}Q, \quad \forall x \in F,$$

$$\bar{\zeta}\eta(x, \bar{x}) + f(x) - f(\bar{x}) = \begin{bmatrix} 10 + 2x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 10 + 2x_1 \\ -3x_2^2 \end{bmatrix} = \begin{bmatrix} 20 + 4x_1 \\ -3x_2^2 \end{bmatrix} \notin -\mathrm{int}Q, \quad \forall x \in F,$$

that is, $\bar{x} \in S_{QI}$ and $\bar{x} \in S_{NQI}$. From Theorem 5.1, one has $\bar{x} \in F^w$. In fact,

$$f(x) - f(\bar{x}) = \begin{bmatrix} 10 + 2x_1 \\ -3x_2^2 \end{bmatrix} \notin -\mathrm{int}Q, \quad \forall x \in F.$$

Theorem 5.2 *Let f be generalized $(-Q)$-invex with respect to $\eta : R^n \times R^n \to R^n$ at $\bar{x} \in F^w$. Then $\bar{x} \in S_{QI}$. Furthermore, assume that f is generalized Q-invex with respect to η at $\bar{x}$. Then $\bar{x} \in S_{NQI}$.*

Proof Let $\bar{x} \in F^w$. Then we have

$$f(x) - f(\bar{x}) \notin -\mathrm{int}Q, \quad \forall x \in F. \tag{7}$$

Since f is generalized $(-Q)$-invex with respect to η at $\bar{x}$, we obtain

$$\zeta\eta(x, \bar{x}) - (f(x) - f(\bar{x})) \in Q, \quad \forall \zeta \in \partial f(\bar{x}).$$

Thus it follows from (7) that

$$\zeta\eta(x, \bar{x}) \notin -\mathrm{int}Q, \quad \forall \zeta \in \partial f(\bar{x}),\ x \in F.$$

Moreover, there exists $\bar{\zeta} \in \partial f(\bar{x})$ such that

$$\bar{\zeta}\eta(x, \bar{x}) \notin -\mathrm{int}Q, \quad \forall x \in F. \tag{8}$$

Therefore, we have $\bar{x} \in S_{QI}$. Since f is generalized Q-invex with respect to η at $\bar{x}$, one has

$$f(x) - f(\bar{x}) - \zeta\eta(x, \bar{x}) \in Q, \quad \forall \zeta \in \partial f(\bar{x}).$$

Furthermore, we get

$$\zeta\eta(x, \bar{x}) + f(x) - f(\bar{x}) - 2\zeta\eta(x, \bar{x}) \in Q, \quad \forall \zeta \in \partial f(\bar{x}).$$

Therefore, by (8), there exists $\bar{\zeta} \in \partial f(\bar{x})$ and

$$\bar{\zeta}\eta(x, \bar{x}) + f(x) - f(\bar{x}) \notin -\mathrm{int}Q, \quad \forall x \in F,$$

that is, $\bar{x} \in S_{NQI}$. This completes the proof. □

6 Conclusions

In this paper, a nonsmooth MOP is investigated under generalized invexity. First, the Kuhn–Tucker type optimality conditions for MOP are obtained. Subsequently, the relationships between weakly efficient solutions of MOP and vector-valued saddle points of its Lagrange function are established. Finally, the relations between weakly efficient solutions of MOP and solutions of HSVQI and HSNVQI are also derived under some suitable assumptions. In order to show our presented results, we also give some examples. For future research, we may study the variational inequalities and optimization problems on (Riemannian) manifolds since there is a little result in these aspects.

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Appendix

FKKM Lemma Guu and Li (2009), Wu et al. (2011): Let K be a nonempty subset of R^m, $G : K \to 2^{R^m}$ be a KKM mapping, i.e., for every finite subset $\{x_1, x_2, \ldots, x_m\}$ of K, co$\{x_1, x_2, \ldots, x_m\}$ is contained in $\bigcup_{i=1}^m G(x_i)$ where co denotes the convex hull, such that for any $x \in K$, $G(x)$ is closed and $G(x^*)$ is bounded for some $x^* \in K$. Then there exists $y^* \in K$ such that $y^* \in G(x)$ for all $x \in K$, i.e., $\bigcap_{x \in K} G(x) \neq \emptyset$.

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