A PENALTY FUNCTION METHOD FOR SOLVING ILL-POSED BILEVEL PROGRAMMING PROBLEM VIA WEIGHTED SUMMATION*

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DOI: 10.1007/s11424-013-2248-5
Received: 15 November 2012 / Revised: 13 June 2013

Abstract For ill-posed bilevel programming problem, the optimistic solution is always the best decision for the upper level but it is not always the best choice for both levels if the authors consider the model's satisfactory degree in application. To acquire a more satisfying solution than the optimistic one to realize the two levels' most profits, this paper considers both levels' satisfactory degree and constructs a minimization problem of the two objective functions by weighted summation. Then, using the duality gap of the lower level as the penalty function, the authors transfer these two levels problem to a single one and propose a corresponding algorithm. Finally, the authors give an example to show a more satisfying solution than the optimistic solution can be achieved by this algorithm.

Key words Bilevel programming, duality gap, penalty function, satisfactory degree, weighted summation.

1 Introduction

Bilevel programming (BLP) problem can be viewed as a static version of the two-player games introduced by Von Stackelberg in the context of unbalanced economic markets. Generally speaking, it is a hierarchical optimization problem. The upper level decision maker (leader) makes decision first and thereafter the lower level decision maker (follower) choose his strategy according to the leader's action. Each decision maker independently seeks its own interest, but is affected by the action of the other decision maker.

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*This research is supported by the National Science Foundation of China under Grant No. 71171150 and the National Natural Science Foundation of China, Tian Yuan Foundation under Grant No. 11226226.
◇ *This paper was recommended for publication by Editor DAI Yuhong.*

If lower level's solution is not a singleton for at least one choice from the upper level, we call this bilevel programming problem ill-posed bilevel programming problem (IBLP). IBLP has many applications in our reality life such as transportation, economics, ecology, engineering, and others. For solving such ill-posed bilevel programming problem, two extreme possibilities have been already considered. The first one, called optimistic model[1−4], assumed the follower would choose the one that favors the leader's interest most, that is full cooperation from the follower to the leader. The second possibility, called pessimistic model[5−8], assumed that the follower will select the worst choice for the leader, that is non-cooperation from the follower to leader. Although the two extreme possibilities have been studied for many years, we always omit considering whether these two models are efficient in application. Even for the optimistic model whose optimal solution is best for the leader, we always omit considering whether the follower is satisfied with this optimistic solution and will cooperate the leader fully in application. We even omit to prove whether the optimistic solution is the most satisfying choice to realize the two levels' most profits. The following Example 1.1[9] is illustrated to show the optimistic solution is not a most satisfying solution for the bilevel programming problem itself.

Example 1.1[9]

$$\begin{aligned}
\max_{x}\ & F_1(x,y) = 8x_1 + 6x_2 + 25y_1 + 30y_2 - 2y_3 - 16y_4 \\
\text{s.t.}\ & x_1 + x_2 \leq 10, \\
& x_1, x_2 \geq 0, \\
\text{where } & y = (y_1, y_2, y_3, y_4)^{\mathrm{T}} \text{ solves,} \\
\max_{y}\ & F_2(x,y) = 10y_1 + 10y_2 + 10y_3 + 10y_4 \\
\text{s.t.}\ & y_1 + y_2 + y_3 + y_4 \leq 10 - x_1 - x_2, \\
& -y_1 + y_4 \leq 0.8x_1 + 0.8x_2, \\
& y_2 + y_4 \leq 4x_2, \\
& y \geq 0.
\end{aligned}$$

The following table 1 shows the results and model's satisfactory degree of optimistic method[9] and fuzzy interactive method[10].

Table 1 Models' satisfactory degree for optimistic solution and the solution to fuzzy interactive method

	(F_1, F_2)	$\mu(F_1)$	$\mu(F_2)$	D
Optimistic method[9]	(252,80)	1	0.8	0.2
Fuzzy interactive method[10]	(251.19, 88.16)	0.997	0.882	0.1180

By the following Definitions 2.1 and 2.2, Table 1 shows the optimistic solution is not the best choice for the model itself because the model's satisfactory degree for the optimistic solution is larger than the solution to fuzzy interactive method[10].

For the optimistic model, many articles have been published these years. In 2007[11], Wang used an adaptive genetic algorithm to solve the IBLP to acquire a globally optimal solution but the leader's objective function is limited to be quadratic. In 2009[12], using the KKT optimality condition of the lower level problem, Lü, et al. transferred a class of IBLP whose lower level is a linear optimization problem to a single level programming with the complementary and slackness condition as a penalty and got an optimal solution. Wan, et al.[13] also achieved a globally optimal solution by a new dual-relax penalty function in 2011. By using the duality gap of the linear lower level problem as a penalty parameter, a global solution is acquired by Zheng, et al. in 2012[14]. Although we can achieve an optimistic solution by these methods, we can't make sure such optimistic solution is also the most satisfactory solution. To acquire a more satisfying solution in application, this paper addresses the class of ill-posed bilevel programming problem in which lower level is linear and also transfers the bilevel programming problem to a single level one by penalty function method. The difference is that we consider the satisfactory degree of the lower lever and put the follower's objective function to the upper level, using the weighted summation, we develop a new algorithm to acquire a sequence solutions by different weight. Where, the penalty method is motivated from Zheng[14,15], but we can achieve a solution whose satisfactory degree for the both levels is no bigger than the optimistic solution.

The rest of this paper is organized as follows. Basic knowledge is introduced in Section 2. In Section 3, we construct a model for this nonlinear bilevel programming problem and establish main results. A new algorithm is given in Section 4 and an example is illustrated to demonstrate the feasibility of the proposed method in Section 5. Finally in Section 6, we finish this paper with a conclusion.

2 Basic Notions in IBLP

In this paper, we consider the following IBLP:

$$\begin{aligned}
&\min_{x} \ F(x,y) \\
&\ \text{s.t.} \ \ x \in X, \\
&\text{where } y \text{ solves}, \\
&\min_{y\geq 0} \ f(x,y) \\
&\ \text{s.t.} \ \ Ax + By \leq b,
\end{aligned} \tag{1}$$

where $x \in R^n$, $y, d \in R^m$, $A \in R^{q\times n}$, $B \in R^{q\times m}$, $b \in R^q$, $F : R^n \times R^m \to R$ is continuous.

Define $Y(x) = \{y \in R^m | By \leq b - Ax, y \geq 0\}$, $S = \{(x,y)|x \in X, y \in Y(x)\}$, $M(x) := \operatorname*{Argmin}_{y} \{f(x,y)|y \in Y(x)\}$.

Definition 2.1 Optimistic model[1−4] (the full cooperation from the follower to the leader) for the above IBLP is:

$$\text{Find } \overline{x} \in X \text{ such that: } \min_{x} \min_{y\in M(x)} F(x,y) = \min_{y\in M(\overline{x})} F(\overline{x},y).$$

Pessimistic model[5−8] (the non-cooperation from the follower to the leader) for IBLP is:

$$\text{Find } \overline{x} \in X \text{ such that: } \min_{x} \max_{y \in M(x)} F(x,y) = \max_{y \in M(\overline{x})} F(\overline{x}, y).$$

Definition 2.2 Membership function $\mu(F)$, $\mu(f)$ are defined as the satisfactory degree of F, f.

$$\mu(F) = \begin{cases} 0, & \text{if} \quad F > F^U, \\ \dfrac{F^U - F}{F^U - F^L}, & \text{if} \quad F^L < F \leq F^U, \\ 1, & \text{if} \quad F < F^L, \end{cases}$$

$$\mu(f) = \begin{cases} 0, & \text{if} \quad f > f^U, \\ \dfrac{f^U - f}{f^U - f^L}, & \text{if} \quad f^L < f \leq f^U, \\ 1, & \text{if} \quad f < f^L, \end{cases}$$

where F^U, f^U, and F^L, f^L denote the upper bound and lower bound of the objective function.

Definition 2.3 Define $D(x,y) = \sqrt{[1-\mu(F)]^2 + [1-\mu(f)]^2}$[16] as the model's satisfactory degree. Obviously, more less the D is, more optimal the decisions of the leader and the follower are.

3 Formulation of the Model

In this section, we consider the objective in the lower level and present a new model with two objectives in the upper level for the optimistic model of Problem (1). The following Problem (2) is the new model:

$$\begin{aligned} &\min_{x,y} \begin{pmatrix} F(x,y) \\ f(x,y) = d^{\mathrm{T}} y \end{pmatrix} \\ &\ \text{s.t.} \ \ x \in X, \\ &\text{where } y \text{ solves,} \\ &\min_{y \geq 0} \ f(x,y) = d^{\mathrm{T}} y \\ &\ \text{s.t.} \ \ Ax + By \leq b. \end{aligned} \tag{2}$$

Definition 3.1 Point $(x^*, y^*) \in X \times M(X)$ is a Pareto-optimal solution of Problem (2), if there is not a point $(x,y) \in S$ such that

$$(x,y) \neq (x^*, y^*) \text{ and } F(x,y) \leq F(x^*, y^*), f(x,y) \leq f(x^*, y^*),$$

where, at least one of the two inequalities is strictly satisfied.

Definition 3.2 A point $(x^*, y^*) \in S$ is said to be a weakly efficient solution of Problem (2) if there is no $(x, y) \in S$ satisfying $F(x, y) < F(x^*, y^*)$ and $f(x, y) < f(x^*, y^*)$.

Remark Here, from the definition 3.1, the optimal solution of Problem (1) must be a Pareto-solution of Problem (2). Problem (2)'s weakly efficient solution must be a Pareto-solution too. And, Problem (2) considers the two objectives of the model and provide more information to acquire a more satisfying solution for not only the upper level but also the lower level.

Then, based on a scalarization technique by means of the weighted summation, we construct a minimization problem of the weighted summation of the objective functions as following Problem (3). $0 \le \lambda \le 1$ is the weight.

$$\begin{aligned}
&\min_{x,y}\ \lambda F(x, y) + (1 - \lambda) f(x, y)\\
&\text{s.t.}\ \ x \in X,\\
&\text{where } y \text{ solves},\\
&\min_{y \ge 0}\ f(x, y) = d^{\mathrm{T}} y\\
&\text{s.t.}\ \ Ax + By \le b.
\end{aligned} \tag{3}$$

Lemma 3.3[17] *If $(x^*, y^*) \in X \times M(X)$ is an optimal solution of Problem (3) for some weight $0 \le \lambda \le 1$, then (x^*, y^*) is a weakly efficient solution of Problem (2).*

Proof Assumed (x^*, y^*) is not a weakly efficient solution of Problem (2). Then, there exists a point $(\overline{x}, \overline{y}) \in S$ satisfying

$$F(\overline{x}, \overline{y}) < F(x^*, y^*),\ \ f(\overline{x}, \overline{y}) < f(x^*, y^*).$$

Then, for a fixed λ satisfies $0 \le \lambda \le 1$, we have

$$\lambda F(\overline{x}, \overline{y}) + (1 - \lambda) f(\overline{x}, \overline{y}) < \lambda F(x^*, y^*) + (1 - \lambda) f(x^*, y^*),$$

which is contradict to (x^*, y^*) is an optimal solution of Problem (3). ∎

Then, for the lower level of Problem (3), using the duality theory of linear programming, the duality problem is:

$$\begin{aligned}
&\max_{\mu \ge 0}\ (Ax - b)^{\mathrm{T}} \mu\\
&\text{s.t.}\ \ -B^{\mathrm{T}} \mu \le d,
\end{aligned} \tag{4}$$

where $U = \{\mu | -B^{\mathrm{T}} \mu \le d, \mu \ge 0\}$, $\pi(x, y, \mu) = d^{\mathrm{T}} y + (b - Ax)^{\mathrm{T}} \mu$ is the duality gap of (4) and the lower level of Problem (3).

Then, we transfer (3) to a single level problem.

$$\begin{aligned}
&\min_{x,y,\mu}\ \lambda F(x, y) + (1 - \lambda) f(x, y)\\
&\text{s.t.}\ \ \pi(x, y, \mu) = 0,\\
&\qquad (x, y) \in S,\ \ u \in U.
\end{aligned} \tag{5}$$

Obviously, Problems (3) and (5) have the same optimal solution.

Finally, using the duality gap as a penalty function, Problem (5) can be transferred as the follows.

$$\begin{aligned} \min_{x,y,\mu} \ & \lambda F(x,y) + (1-\lambda)f(x,y) + k\pi(x,y,\mu) \\ \text{s.t.} \ & (x,y) \in S, \ u \in U, \end{aligned} \tag{6}$$

where k is the penalty parameter, $\lambda = 1$ is the optimistic model for IBLP by using penalty function method.

Lemma 3.4 *If* (x_k, y_k, μ_k) *is the optimal solution to Problem* (6) *and it is a feasible solution to Problem* (5), *then* (x_k, y_k) *is the optimal solution to Problem* (5).

Theorem 3.5[14] *Assuming the following conditions are satisfied, there exists an optimal solution* $(x^*, y^*, \mu^*) \in S \times E(U)$ *for Problem* (6), *where* $E(U)$ *is a set for all vertex point of* U.

Condition 1 For any $x \in X$, there exists a compact subset Z such that $Y(x) \neq \emptyset$, $Y(x) \subset Z$.

Condition 2 $X \neq \emptyset$ and X is compact.

Proof For a fixed $k > 0$, $\mu \in U$, $0 < \mu < 1$, define

$$\theta_k(\mu) = \min_{(x,y)\in S} \{\lambda F(x,y) + (1-\lambda)f(x,y) + k[d^{\mathrm{T}}y + (b - Ax)^{\mathrm{T}}\mu]\}.$$

Because $\lambda F(x,y) + (1-\lambda)f(x,y)$ is continuous in S, $\theta_k(\mu)$ is concave, and the following:

$$\min_{\mu\in U} \ \theta_k(\mu) \geq \min_{(x,y)\in S} \ \lambda F(x,y) + (1-\lambda)f(x,y).$$

By Lemma 32.3.4 of Reference [18], there exists an optimal solution $\mu^* \in E(U)$ for problem of $\min_{\mu\in U} \ \theta_k(\mu)$.

Then define

$$\theta_k(\mu^*) = \min_{(x,y)\in S} \{\lambda F(x,y) + (1-\lambda)f(x,y) + k[d^{\mathrm{T}}y + (b - Ax)^{\mathrm{T}}\mu^*]\}. \tag{7}$$

Because $\lambda F(x,y) + (1-\lambda)f(x,y)$ is continuous in S, there exists an optimal solution (x^*, y^*) for Problem (7).

All above, there exists an optimal solution $(x^*, y^*, \mu^*) \in S \times E(U)$ for Problem (6). ∎

Theorem 3.6[14] *Assume Conditions* 1 *and* 2 *are satisfied,* $\{(x_k, y_k, \mu_k)\}$ *is a sequence of optimal solutions to Problem* (6). *Then, there exists* $k > 0$ *such that* $\pi(x_k, y_k, \mu_k) = 0$.

Proof By Theorem 3.5, there exists an solution to Problem (6). Then, Problem (5) also have solutions. Assume $(\overline{x}, \overline{y}, \overline{\mu})$ is an optimal solution to Problem (5), we have $\pi(\overline{x}, \overline{y}, \overline{\mu}) = 0$.

$$\begin{aligned} & \lambda F(x_k, y_k) + (1-\lambda)f(x_k, y_k) + k\pi(x_k, y_k, \mu_k) \\ \leq \ & \lambda F(\overline{x}, \overline{y}) + (1-\lambda)f(\overline{x}, \overline{y}) + k\pi(\overline{x}, \overline{y}, \overline{\mu}) \\ = \ & \lambda F(\overline{x}, \overline{y}) + (1-\lambda)f(\overline{x}, \overline{y}). \end{aligned}$$

Then, $k\pi(x_k, y_k, \mu_k) \leq \lambda F(\overline{x}, \overline{y}) + (1-\lambda)f(\overline{x}, \overline{y}) - \{\lambda F(x_k, y_k) + (1-\lambda)f(x_k, y_k)\}$.

Because $\lambda F(x,y)+(1-\lambda)f(x,y)$ is continuous in S, there exists a constant $M>0$ such that:

$$0 \le k\pi(x_k, y_k, \mu_k) \le M.$$

Then, for k is big enough, we have: $\pi(x_k, y_k, \mu_k) = 0$. ∎

Theorem 3.7[14] *Assume Conditions* 1 *and* 2 *are satisfied,* $\{(x_k, y_k, \mu_k)\}$ *is a sequence of optimal solutions to Problem* (6)*. If there exists* $\overline{k} > 0$ *such that* $\pi(x_{\overline{k}}, y_{\overline{k}}, \mu_{\overline{k}}) = 0$*, then for all* $k > \overline{k}$*, we have:* $\pi(x_k, y_k, \mu_k) = 0$.

Proof For any $k > \overline{k}$, $0 < \lambda < 1$, we have:

$$\begin{aligned}\lambda F(x_{\overline{k}}, y_{\overline{k}}) + (1-\lambda) f(x_{\overline{k}}, y_{\overline{k}}) &= \lambda F(x_{\overline{k}}, y_{\overline{k}}) + (1-\lambda) f(x_{\overline{k}}, y_{\overline{k}}) + \overline{k}\pi(x_{\overline{k}}, y_{\overline{k}}, \mu_{\overline{k}})\\ &\le \lambda F(x_k, y_k) + (1-\lambda) f(x_k, y_k) + \overline{k}\pi(x_k, y_k, \mu_k)\\ &\le \lambda F(x_k, y_k) + (1-\lambda) f(x_k, y_k) + k\pi(x_k, y_k, \mu_k)\\ &\le \lambda F(x_{\overline{k}}, y_{\overline{k}}) + (1-\lambda) f(x_{\overline{k}}, y_{\overline{k}}) + k\pi(x_{\overline{k}}, y_{\overline{k}}, \mu_{\overline{k}})\\ &= \lambda F(x_{\overline{k}}, y_{\overline{k}}) + (1-\lambda) f(x_{\overline{k}}, y_{\overline{k}}).\end{aligned}$$

So, if $k > \overline{k}$, we have: $\pi(x_k, y_k, \mu_k) = 0$. ∎

Next, we will develop an algorithm to solve Problem (6).

4 Algorithm Based on Satisfactory Degree via Penalty Function

In this section, we will acquire a sequence of solutions by different weight λ, then, computing the satisfactory degree, we choose the least degree as the outcome.

Algorithm 4.1

Step 1 Set $k > 0$, $\delta > 1$, $\eta > 0$, $\lambda \ge 0$, $V = \emptyset$, $t = 0$.

Step 2 Compute all the vertex of polyhedron U. Let $E(U) = \{\mu_1, \mu_2, \cdots, \mu_p\}$.

Step 3 Compute the following problem $P(\mu_i)$:

$$\min_{(x,y)\in S} \ \lambda F(x,y) + (1-\lambda) f(x,y) + k\pi(x, y, \mu_i).$$

Set the optimal solution is (x_i, y_i).

Step 4 Set $\theta(\mu^*) = \min\{\theta(\mu_i) | 1 \le i \le p\}$, and (x_t^*, y_t^*) is the optimal solution of $P(\mu^*)$. Where, $\theta(\mu_i)$ is the optimal value of Problem $P(\mu_i)$.

Step 5 If $\pi(x^*, y^*, \mu^*) = 0$, computing $D(x_t^*, y_t^*)$, set $V = V \cup D(x_t^*, y_t^*)$, $t := t+1$, $\lambda = \lambda + \eta$ and go to Step 3; else, set $k = \delta k$ and go to Step 1. If $\lambda > 1$, go to Step 6.

Step 6 Computing minimal value of set V, the corresponding (x_t^*, y_t^*) is the best solution and stop.

5 Numerical Example

To illustrate the feasibility of the proposed method, we consider the following example:

Example 5.1

$$\begin{aligned}
&\min_{x} \ x^2 - 5x + 6y_1 + 5y_2 \\
&\text{s.t.} \ \ 0 \le x \le 3, \\
&\text{where } y \text{ solves,} \\
&\min_{y} \ -0.5y_2 + y_3 + 2y_4 \\
&\text{s.t.} \ \ 3 \ge y_i \ge 0, \\
&\qquad -0.1x - y_1 - y_2 \le -1, \\
&\qquad 0.2x + 1.25y_2 - y_4 \le -1, \\
&\qquad -x + 6y_1 + y_2 - 2y_3 \le 1.
\end{aligned}$$

Where, compute the upper bound and lower bound of the objective function F and f, we have: $F^U = 4.5$, $F_L = -6.25$, $f^U = 9$, and $f_L = 2$. The vertex of polyhedron U is $\mu = (2.5, 2, 0.5)^{\mathrm{T}}$. Set the weight equals to 0.1,0.2, 0.5, 0.7, 1, respectively, we have the following result in Table 2.

Table 2 Computational results of Example 5.1

λ	x	y_1	y_2	y_3	y_4	D
0.1	0.75	0.1650	0.7600	0.0000	2.0901	0.404583077
0.2	1.625	0.0000	0.8934	0.0026	2.3559	0.343443779
0.5	2.25	0.6688	0.8178	0.0001	2.1357	0.768555307
0.7	2.321	0.8285	0.0016	0.0000	1.2455	0.471374055
1	2.325	0.6394	0.1281	0.0000	1.6252	0.344261037

From Table 2, it is the optimistic model when $\lambda = 1$, but its solution's satisfactory degree is larger than the solution of $\lambda = 0.2$. That is, if we choose $\lambda = 0.2$, we can acquire a more satisfying result for both upper level and lower level.

6 Conclusions

In this paper, we consider both levels' profits and construct a minimization Problem of the weighted summation of the objective functions based on the scalarization technique. Then, using the duality gap as the penalty function and the satisfactory degree as a rule, we give an algorithm to acquire a more satisfying solution for IBLP than the optimistic solution. The illustrative example demonstrates the feasibility of the proposed Problem and the algorithm.

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