On Generalized Mixed Equilibrium Problems and Fixed Point Problems with Applications

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Abstract: In this paper, we introduce and investigate two new generalized mixed equilibrium problems and explore the relationship between them and the properties of their solutions in Banach spaces. Based on the generalized f-projection, we construct hybrid algorithms to find common fixed points of a countable family of quasi-ϕ-nonexpansive mappings in Banach spaces, a common element of the set of solutions of generalized mixed equilibrium problems and the set of fixed points for quasi-ϕ-nonexpansive mappings and, further, prove some strong convergence theorems for these hybrid algorithms under some suitable assumptions. As some applications of the main results, the strong convergence theorems for the general H-monotone mappings and equilibrium problems are also proven.

Key–Words: Generalized mixed equilibrium problem, Quasi-ϕ-nonexpansive mapping, Strong convergence theorem, Fixed point, Generalized f-projection.

1 Introduction

Let E be a real Banach space with the dual space E^*, C be a nonempty closed convex subset of E. The norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

In 1994, Alber [1] introduced the generalized projections $\pi_C : E^* \to C$ and $\Pi_C : E \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties. Furthermore, he applied the generalized projections to approximately solving variational inequalities and Von Neumann intersection problem in Banach spaces. Li [3] extended the generalized projection from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and established a Mann type iterative scheme for finding the approximate solutions for the classical variational inequality problem in compact subsets of Banach spaces. Wu and Huang [4] introduced a generalized f-projection $\pi_C^f : E^* \to 2^C$, which extended the generalized projection operator, and proved some properties of the generalized f-projection operator in Banach spaces. In addition, they showed an interesting relation between the generalized f-projection operator and the resolvent operator for the subdifferential of a proper convex and lower semicontinuous function in reflexive and smooth Banach spaces and proved the generalized f-projection operator is maximal monotone in [5]. Fan, Liu and Li[6] presented some basic results for the generalized f-projection operator and discussed the existence of solutions and approximation of solutions for generalized variational inequalities in noncompact subsets of Banach spaces by the iterative schemes. Recently, Li, Huang, O'Regan [7] introduced another generalized f-projection operator and gave some properties of this projection and proved strong convergence theorems for relatively nonexpansive mapping in Banach spaces. As applications, they also proved some strong convergence theorems for H-monotone mappings in Banach spaces.

Let $\Theta : C\times C \to R$ be a bifunction, $\psi : C \to R$ be a real-valued function, $A : C \to E^*$ be a nonlinear mapping and $\eta : C \times C \to E$ be a mapping. We consider the so-called *generalized mixed equilibrium problem* (GMEP1): Find $x \in C$ such that

$$\Theta(x,y) + \langle A(x), \eta(y,x)\rangle + \psi(y) - \psi(x) \geq 0 \quad (1)$$

for all $y \in C$. Denote the set of solutions to (1) by Ω_1.

Special cases of the problem (1) are as follows:

(I) If $\eta(y,x) = y-x$ for all $x,y \in C$, the problem (1) is equivalent to the problem: Find $x \in C$ such that

$$\Theta(x,y) + \langle A(x), y-x\rangle + \psi(y) - \psi(x) \geq 0 \quad (2)$$

for all $y \in C$, which is called the *generalized mixed equilibrium problem* [8]. The set of solutions to (2) is denoted by (GMEP).

(II) If $A(x) = 0$ for all $x \in C$, the problem (1) is equivalent to the problem: Find $x \in C$ such that

$$\Theta(x, y) + \psi(y) - \psi(x) \geq 0 \tag{3}$$

for all $y \in C$, which is called the *mixed equilibrium problem* [9]. The set of solutions to (3) is denoted by (MEP).

(III) If $\Theta(x, y) = 0$ for all $x, y \in C$, the problem (1) is equivalent to the problem: Find $x \in C$ such that

$$\langle A(x), \eta(y, x)\rangle + \psi(y) - \psi(x) \geq 0 \tag{4}$$

for all $y \in C$, which is called the *mixed quasi-variational inequality problem*. The set of solutions to (4) is denoted by (MQVIP).

(IV) If $\Theta(x, y) = 0, \eta(y, x) = y - x$ for all $x, y \in C$, the problem (1) is equivalent to the problem: Find $x \in C$ such that

$$\langle A(x), y - x\rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C, \tag{5}$$

for all $y \in C$, which is called the *mixed variational inequality problem* of Browder type [15]. The set of solutions to (5) is denoted by (MVIP).

(V) If $A(x) = 0, \psi(x) = 0$ for all $x \in C$, the problem (1) is equivalent to the problem: Find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C, \tag{6}$$

which is called the *equilibrium problem* of Blum and Oettli [16]. The set of solutions to (6) is denoted by EP(Θ).

Recently, many authors studied the problems of finding common fixed points of nonexpansive mappings, a common element of the set of fixed points of nonexpansive mappings and the set of solutions of equilibrium problems in the setting of Hilbert spaces and uniformly smooth and uniformly convex Banach spaces, respectively, (see, e.g., [8, 9, 10, 11, 12, 13, 14, 18, 23] and the references therein). Very recently, Takahashi and Zembayashi [19] proved strong and weak convergence theorems for finding a common element of the set of solutions of the equilibrium problem (6) and the set of fixed points of a relatively nonexpansive mapping in Banach spaces as follows:

Theorem TZ [19] *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to R$ satisfy Assumption 2 (see Section 2) and $T : C \to C$ be a relatively nonexpansive mapping such that $EP(\Theta) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in C, \\ y_n = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n)J(Tx_n)), \\ u_n = \{z \in C : \Theta(z, y) \\ +\frac{1}{r_n}\langle y - z, J(z) - J(y_n)\rangle \geq 0, \forall y \in C\}, \\ C_n = \{z \in C : \langle x_n - z, J(x) - J(x_n)\rangle \geq 0\}, \\ Q_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is the normalized duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $a > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{F(T)\cap EP(\Theta)} x_0$.

Inspired and motivated by the researches going on in this direction, first, we introduce and investigate two new generalized mixed equilibrium problems and explore the relationship between them and the properties of their solutions. Secondly, by using the conception of the generalized f-projection [7], we construct hybrid algorithms to find common fixed points of a countable family of quasi-ϕ-nonexpansive mappings, a common element of the set of fixed points of quasi-ϕ-nonexpansive mappings and the set of solutions of generalized mixed equilibrium problems in Banach spaces and, further, we prove the strong convergence of these hybrid algorithms under some suitable assumptions. Finally, as applications of our main results, we discuss some strong convergence theorems for general H-monotone mappings and equilibrium problems.

2 Preliminaries

Throughout this paper, we denote by R the sets of real numbers. Let C be a nonempty closed convex subset of a Banach space E. Let $\eta : C \times C \to E, T : C \to C$ be the mappings with $F(T) = \{x \in C : Tx = x\}$ and $f : E \to R \cup \{+\infty\}$ be proper convex and lower semicontinuous. We denote by $J : E \to 2^{E^*}$ the *normalized duality mapping* defined by

$$J(x) = \{j(x) \in E^* : \langle j(x), x\rangle = \|j(x)\|\|x\| = \|x\|^2\}$$

for all $x \in E$.

Without confusion, one understands that $\|j(x)\|$ is the E^*-norm and $\|x\|$ is the E-norm. Many properties of the normalized duality mapping J can be found in (see, for example, [1, 19, 21, 22, 24, 25]) and we list them as follows:

$(p1)$ $J(x)$ is nonempty for each $x \in E$;

$(p2)$ J is a monotone and bounded operator in Banach space;

$(p3)$ J is a strictly monotone operator in strictly convex Banach space;

$(p4)$ J is the identity operator in Hilbert space;

$(p5)$ If E is a reflexive, smooth and strictly convex Banach space and $J^* : E^* \to 2^E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*, JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mapping on E and E^*, respectively.

$(p6)$ If E is strictly convex Banach space, then J is one to one, i.e.,

$$x \neq y \Rightarrow J(x) \cap J(y) = \emptyset;$$

$(p7)$ If E is smooth, then J is single valued;

$(p8)$ E is uniformly convex Banach space if and only if E^* is uniformly smooth;

$(p9)$ If E is uniformly convex and uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on bounded subsets of E and $J^{-1} = J^*$ is also uniformly norm-to-norm continuous on bounded subsets of E^*.

Let E be a smooth Banach space. Define a function $\phi : E \times E \to R$ as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad \forall x, y \in E,$$

and set

$$G(x, J(y)) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2 + 2\rho f(x)$$

for all $x \in C, y \in E$, where ρ is a positive number and $f : C \to R \cup \{+\infty\}$ is proper convex and lower semicontinuous, respectively.

Remark 1 *[3, 4, 5, 21] We have the following properties of the functions ϕ and G:*

(1) If $C = E$ and $f(x) = 0$ for all $x \in C$, then $G(x, y) = \phi(x, y)$ for all $x, y \in C$;

(2) If E is a reflexive strictly convex and smooth Banach space, then, for all $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$;

(3) If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$;

(4) For all $x, y \in E$, $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$.

Notation: " $\rightharpoonup$ " stands for weak convergence and " $\to$ " for strong convergence.

We first recall some definitions and lemmas for the main results in this paper.

Assumption 2 *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $\Theta : C \times C \to R$ be a mapping satisfying the following conditions:*

$(C1)$ $\Theta(x, x) = 0$ for all $x \in C$;

$(C2)$ Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

$(C3)$ Θ is upper-hemicontinuous, i.e., for all $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(x + t(z - x), y) \leq \Theta(x, y);$$

$(C4)$ for all $x \in C$, $\Theta(x, \cdot)$ is convex and lower semicontinuous.

Definition 3 *A point $p \in C$ is called an asymptotic fixed point of T if there exists a sequence $\{x_n\} \subset C$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

We denote the set of asymptotic fixed points of T by $\tilde{F}(T)$.

Definition 4 *[7] Let C be a nonempty closed and convex subset of a smooth Banach space E. An operator $\Pi_C^f : E \to 2^C$ is called a generalized f-projection if*

$$\Pi_C^f(y) = \{z \in C : G(z, J(y)) \leq G(x, J(y)),\ \forall x \in C\}$$

for all $y \in E$.

Remark 5 *If $f(x) = 0$ for all $x \in C$, then the generalized f-operator $\Pi_C^f(y)$ reduces to the generalized operator $\Pi_C(y)$.*

Remark 6 *If E is a Hilbert space and $f(x) = 0$ for all $x \in C$, then the generalized f-operator $\Pi_C^f(y)$ and generalized operator $\Pi_C(y)$ are reduced to the following metric projection operator*

$$P_C(y) = \{z \in C : \|z - y\| \leq \|x - y\|,\ \forall x \in C\}.$$

The following example illustrates that the extension of metric projection operator is nontrivial.

Example 7 *[2, 17] Let $\rho = 1, E = R^3$ with the norm $\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2} + \sqrt{x_2^2 + x_3^2}$ and $C = \{x = (x_1, x_2, x_3) \in E : x_2 = x_3 = 0\}$. Define $f : C \to R \cup +\infty$ by*

$$f(x) = \begin{cases} 2 + 2\sqrt{5}, & \text{if} \quad x_1 < 0, \\ -2 - 2\sqrt{5}, & \text{otherwise}. \end{cases}$$

Then E is a smooth strictly convex Banach space. For $x_0 = (1, 1, 1) \in E$, simple computation allows that $P_C(x_0) = (1, 0, 0), \Pi_C(x_0) = (2, 0, 0)$ and $\Pi_C^f(x_0) = (4, 0, 0)$. □

Definition 8 *[21] Let $S : E \to C$ be a mapping. S is called a firmly nonexpansive-type mapping if*

$$\begin{aligned}&\phi(Sx, Sy) + \phi(Sy, Sx)\\ \leq\ &\phi(Sx, y) + \phi(Sy, x) - \phi(Sx, x) - \phi(Sy, y)\end{aligned}$$

for all $x, y \in E$.

From Definition 8, we have the following:

Proposition 9 *Let $S : E \to C$ be a firmly nonexpansive-type mapping such that $F(S) \neq \emptyset$. Then $G(p, J(Sx)) + \phi(Sx, x) \leq G(p, J(x))$ and $G(p, J(Sx)) \leq G(p, J(x))$ for all $p \in F(S), x \in E$.*

Definition 10 *[8, 26] Let C be a nonempty closed and convex subset of a smooth Banach space and $T : C \to C$ be a mapping. T is called:*

(1) *a quasi-ϕ-nonexpansive mapping if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;*

(2) *a relatively nonexpansive mapping if $\tilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;*

(3) *a nonexpansive mapping if*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(4) *closed if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.*

Remark 11 *[7, 26]* (1) *It is easy to see that a quasi-ϕ-nonexpansive mapping T is equivalent to $F(T) \neq \emptyset$ and $G(p, J(Tx)) \leq G(p, J(x))$ for all $x \in C$ and $p \in F(T)$;*

(2) *Every relatively nonexpansive mapping is closed and quasi-ϕ-nonexpansive mapping;*

(3) *If E is a Hilbert space, then a nonexpansive mapping is a relatively nonexpansive mapping.*

In [13], Qin, Cho and Kang gave the following example which is closed and quasi-ϕ-nonexpansive.

Example 12 *Let E be a uniformly smooth and strictly convex Banach space and $M : E \to 2^{E^*}$ be a maximal monotone mapping with $M^{-1}(0) \neq \emptyset$. Then, the resolvent operator $J_\lambda = (J + \lambda M)^{-1}J$ is a closed quasi-ϕ-nonexpansive mapping from E onto $D(M)$ and $F(J_\lambda) = M^{-1}(0)$, where $\lambda > 0$. Moreover, J_λ is also a relatively nonexpansive mapping.* □

Definition 13 *[27] Let E be a Banach space with the dual space E^* and C be a nonempty subset of E. Let $A : C \to E^*$ and $\eta : C \times C \to E$ be two mappings. The mapping A is said to be:*

(1) *η-hemicontinuous if, for any given $x, y \in C$, the function $q : [0, 1] \to R$ is defined by*

$$q(t) = \langle A((1 - t)x + ty), \eta(y, x)\rangle$$

is continuous at 0^+;

(2) *η-monotone if $\langle A(x) - A(y), \eta(x, y)\rangle \geq 0$ for all $x, y \in C$.*

Lemma 14 *[28] Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 15 *[7] Let C be a nonempty closed convex subset of a smooth and reflexive Banach space E and let $y \in E$. Then*

$$\phi(x, \Pi_C^f(y)) + G(\Pi_C^f(y), J(y)) \leq G(x, J(y))$$

for all $x \in C$.

Lemma 16 *[29] Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing continuous and convex function $h : [0, 2r] \to R$ such that $h(0) = 0$ and*

$$\Big\|\sum_{i=1}^{\infty} \alpha_i x_i\Big\|^2 \leq \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \alpha_1\alpha_k h(\|x_1 - x_k\|)$$

for all $k \in N$, $\{x_i\}_{i=1}^\infty \subset B_r$ and $\{\alpha_i\}_{i=1}^\infty \subset [0, 1]$ with $\sum_{i=1}^\infty \alpha_i = 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 17 *[16] Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space E, $\Theta : C \times C \to R$ satisfy the conditions $(C1)$-$(C4)$ of Assumption 2 and let $r > 0$, $x \in E$. Then there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, J(z) - J(x)\rangle \geq 0, \quad \forall y \in C.$$

Lemma 18 *[19] Let C be a nonempty closed convex subset of a uniformly smooth strictly convex and reflexive Banach space E and $\Theta : C \times C \to R$ satisfy Assumption 2. For any $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ by*

$$\begin{aligned}T_r(x) = \quad \{ \quad &z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z,\\ &J(z) - J(x)\rangle \geq 0, \forall y \in C\}\end{aligned}$$

for all $x \in E$. Then the following statements hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping;*

(3) *$F(T_r) = EP(\Theta)$ and $EP(\Theta)$ is closed and convex.*

Lemma 19 *[19] Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space E and $T : C \to C$ be a quasi-ϕ-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 20 *[30] Let $f : E \to R \cup \{+\infty\}$ be proper convex and lower semicontinuous. Then there exist $y^* \in E^*$ and $\alpha \in R$ such that*

$$f(y) \geq \langle y, y^* \rangle + \alpha, \quad \forall y \in E.$$

Lemma 21 *[19] Let E be a smooth uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing continuous and convex function $h : [0, 2r] \to R$ such that $h(0) = 0$ and $h(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.*

Lemma 22 *[7] Let C be a nonempty closed convex subset of a real smooth and reflexive Banach space E. Then the following statements hold:*

(1) $\Pi_C^f(x)$ is nonempty closed convex subset of C for all $x \in E$;

(2) For all $x \in E$, $\hat{x} \in \Pi_C^f(x)$ if and only if

$$\langle \hat{x} - y, J(x) - J(\hat{x}) \rangle + \rho f(y) - \rho f(x) \geq 0, \forall y \in C;$$

(3) If E is strictly convex, then Π_C^f is a single valued mapping.

3 Generalized Mixed Equilibrium Problems

In this section, we investigate the relationship between (GMEP1) and the following generalized mixed equilibrium problem (GMEP2) and the properties of their solutions in a Banach space under some suitable conditions. (GMEP2): Find $x \in C$ such that

$$\Theta(x, y) + \langle A(y), \eta(y, x) \rangle + \psi(y) - \psi(x) \geq 0$$

for all $y \in C$.

Denote the set of solutions of (GMEP2) by Ω_2.

Theorem 23 *Let C be a nonempty closed convex subset of a Banach space E. Let $\Theta : C \times C \to R$ satisfy the condition $(C4)$ of Assumption 2, $\eta : C \times C \to E$, $A : C \to E^*$ be an η-hemicontinuous and η-monotone mapping and $\psi : C \to R$ be a convex function. Assume that*

(a) $\eta(y, y) = 0$ for all $y \in C$;

(b) for any $u, v \in C$, the mapping $x \mapsto \langle A(v), \eta(x, u) \rangle$ is convex.

Then $(GMQEP1)$ and $(GMQEP2)$ are equivalent, i.e., $\Omega_1 = \Omega_2$.

Proof: Let $\bar{x} \in \Omega_1$. Then, for all $y \in C$,

$$\Theta(\bar{x}, y) + \langle A(\bar{x}), \eta(y, \bar{x}) \rangle + \psi(y) - \psi(\bar{x}) \geq 0. \quad (7)$$

Since A is an η-monotone mapping, we get

$$\langle A(y) - A(\bar{x}), \eta(y, \bar{x}) \rangle \geq 0, \quad \forall y \in C,$$

and so

$$\langle A(y), \eta(y, \bar{x}) \rangle \geq \langle A(\bar{x}), \eta(y, \bar{x}) \rangle, \quad \forall y \in C.$$

Therefore, from (7), it follows that

$$\Theta(\bar{x}, y) + \langle A(y), \eta(y, \bar{x}) \rangle + \psi(y) - \psi(\bar{x}) \geq 0$$

for all $y \in C$. This means that $\bar{x} \in \Omega_2$.

Conversely, let $\bar{x} \in \Omega_2$. Then

$$\Theta(\bar{x}, y) + \langle A(y), \eta(y, \bar{x}) \rangle + \psi(y) - \psi(\bar{x}) \geq 0 \quad (8)$$

for all $y \in C$. For any $y \in C$, put $y_t = (1 - t)\bar{x} + ty$ for all $t \in (0, 1)$. Then $y_t \in C$. From (8), it follows that

$$\Theta(\bar{x}, y_t) + \langle A(y_t), \eta(y_t, \bar{x}) \rangle + \psi(y_t) - \psi(\bar{x}) \geq 0. \quad (9)$$

Noticing that

$$\Theta(\bar{x}, y_t) \leq (1 - t)\Theta(\bar{x}, \bar{x}) + t\Theta(\bar{x}, y) = t\Theta(\bar{x}, y),$$

$$\psi(y_t) \leq (1 - t)\psi(\bar{x}) + t\psi(y)$$

and

$$\begin{aligned} & \langle A(y_t), \eta(y_t, \bar{x}) \rangle \\ \leq\ & (1 - t)\langle A(y_t), \eta(\bar{x}, \bar{x}) \rangle + t\langle A(y_t), \eta(y, \bar{x}) \rangle \\ =\ & t\langle A(y_t), \eta(y, \bar{x}) \rangle. \end{aligned}$$

Again, from (9), it follows that

$$t\Theta(\bar{x}, y) + t\langle A(y_t), \eta(y, \bar{x}) \rangle + t\psi(y) - t\psi(\bar{x}) \geq 0.$$

Since $t \in (0, 1)$, we obtain

$$\Theta(\bar{x}, y) + \langle A(y_t), \eta(y, \bar{x}) \rangle + \psi(y) - \psi(\bar{x}) \geq 0.$$

By the η-hemicontinuity of A, we have

$$\Theta(\bar{x}, y) + \langle A(\bar{x}), \eta(y, \bar{x}) \rangle + \psi(y) - \psi(\bar{x}) \geq 0.$$

Therefore, we have $\bar{x} \in \Omega_1$. The proof is then complete. □

Theorem 24 *Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space E. Let $\Theta : C \times C \to R$ be a bifunction satisfying the conditions $(C1)$-$(C4)$ of Assumption 2, $\eta : C\times C \to E$, $A : C \to E^*$ be an η-hemicontinuous and η-monotone mapping, $\psi : C \to R$ be a convex lower semicontinuous function and let $r > 0$, $z \in E$. Assume that*

(a) $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;

(b) for any $v \in C$, the mapping $x \mapsto \langle A(x), \eta(v, x)\rangle$ is convex and lower semicontinuous. Then the following statements hold:

(I) There exists $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) + \langle A(\bar{x}), \eta(y, \bar{x})\rangle + \psi(y) - \psi(\bar{x}) + \frac{1}{r}\langle y - \bar{x}, J(\bar{x}) - J(z)\rangle \geq 0, \quad \forall y \in C.$$

(II) If we define a mapping $U_r : E \to C$ by, for any $z \in E$,

$$U_r(z) = \{u : \Theta(u, y) + \langle A(u), \eta(y, u)\rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, J(u) - J(z)\rangle \geq 0,\ \forall y \in C\},$$

then the mapping U_r has the following properties:

(1) U_r is single-valued and firmly nonexpansive-type mapping;

(2) $F(U_r) = \Omega_1 = \tilde{F}(U_r)$;

(3) Ω_1 is closed and convex.

Proof: For the sake of simplicity, we define a function $H : C \times C \to R$ by, for all $x, y \in C$,

$$H(x, y) = \Theta(x, y) + \langle A(x), \eta(y, x)\rangle + \psi(y) - \psi(x).$$

Then, for all $z \in E$,

$$U_r(z) = \{u \in C : H(u, y) + \frac{1}{r}\langle y - u, J(u) - J(z)\rangle \geq 0, \forall y \in C\}.$$

From the definition of H, it is easy to see that $H(x, x) = 0$ for all $x \in C$. By the η-monotonicity of A, it follows from $\eta(x, y) + \eta(y, x) = 0$ that

$$H(x, y) + H(y, x) \leq 0, \quad \forall x, y \in C.$$

Since Θ satisfies the conditions $(C3)$ and $(C4)$ of Assumption 2, $\psi : C \to R$ is convex and lower semicontinuous, it follows from (b) that the function $y \mapsto H(x, y)$ is convex and lower semicontinuous and so, for any $x, y, v \in C$,

$$\begin{aligned} & \limsup_{t\to 0^+} H(x + t(v - x), y) \\ = \ & \limsup_{t\to 0^+}[\langle A(x + t(v - x)), \eta(y, x + t(v - x))\rangle \\ & + \Theta(x + t(v - x), y) + \psi(y) \\ & - \psi(x + t(v - x))] \\ \leq \ & \Theta(x, y) + \langle A(x), \eta(y, x)\rangle + \psi(y) - \psi(x) \\ = \ & H(x, y). \end{aligned}$$

Summing up the above arguments, we know that the function H satisfying the conditions $(C1)$-$(C4)$ of Assumption 2. Therefore, from Lemma 18, we can obtain the desired conclusions. This completes the proof. □

From Theorems 23 and 24, we have the following:

Corollary 25 *Suppose that the assumptions of Theorems 23 and 24 hold. Then $\Omega_1 = \Omega_2$ and Ω_1 and Ω_2 are closed and convex.*

4 Main results

In this section, we explore several strong convergence theorems for a countable family of quasi-ϕ-nonexpansive mappings and the generalized mixed equilibrium problems in Banach spaces under some suitable conditions.

Theorem 26 *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, $T_i : C \to C$ for each $i \geq 1$ be a closed and quasi-ϕ-nonexpansive mapping such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $f : E \to R \cup \{+\infty\}$ be a proper convex and lower semicontinuous mapping with lower bound. Define a sequence $\{x_n\}$ in C by the following* **Algorithm I** *:*

$$\begin{cases} x_0 \in C,\ Q_0 = C, \\ y_n = J^{-1}(\beta_{n0}J(x_n) + \sum_{i=1}^{\infty} \beta_{ni}J(T_i x_n)), \\ Q_{n+1} = \{z \in Q_n : G(z, J(y_n)) \leq G(z, J(x_n))\}, \\ x_{n+1} = \Pi^f_{Q_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\{\beta_{ni}\} \subset [0, 1]$ for each $i \geq 1$ such that $\liminf_{n\to\infty} \beta_{n0}\beta_{ni} > 0$ and $\sum_{j=0}^{\infty} \beta_{nj} = 1$. Then the sequence $\{x_n\}$ converges strongly to the point $\Pi^f_{\cap_{i=1}^{\infty} F(T_i)} x_0$.

Proof: By Lemma 19, it follows that $\cap_{i=1}^{\infty} F(T_i)$ is a nonempty closed and convex subset of E. Hence $\Pi^f_{\cap_{i=1}^{\infty} F(T_i)}$ is well-defined. According to $Q_0 = C$, it follows that Q_0 is a nonempty closed convex set and $\cap_{i=1}^{\infty} F(T_i) \subset Q_0$. For each $n \geq 1$, it is easy to check that Q_n for each $n \geq 1$ is closed. For any $z \in Q_{n+1} \subset Q_n$, we have

$$\begin{aligned} & G(z, J(y_n)) \leq G(z, J(x_n)) \\ \Leftrightarrow \ & -2\langle z, J(y_n) - J(x_n)\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0. \end{aligned}$$

Take $v_1, v_2 \in Q_{n+1}$ arbitrarily and put $v_t = tv_1 + (1-t)v_2$ for all $t \in [0,1]$. Since, for each $i \in \{1,2\}$,

$$-2\langle v_i, J(y_n) - J(x_n)\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0,$$

we have

$$-2\langle v_t, J(y_n) - J(x_n)\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0,$$

i.e., $v_t \in Q_{n+1}$. Therefore, Q_{n+1} is closed convex for each $n \geq 0$. This implies that $\{x_n\}$ is well-defined.

Next, we show, by induction, that $\cap_{i=1}^{\infty} F(T_i) \subset Q_n$ for all $n \geq 0$. First, $\cap_{i=1}^{\infty} F(T_i) \subset Q_0$ is clear. Now, suppose that $\cap_{i=1}^{\infty} F(T_i) \subset Q_k$ for some $k \geq 0$. Let $\omega \in \cap_{i=1}^{\infty} F(T_i)$. Then $\omega \in Q_k$. By virtue of Remark 1, we obtain

$$\begin{aligned}
& 2\rho f(\omega) \\
\leq\ & (\|y_n\| - \|\omega\|)^2 + 2\rho f(\omega) \\
\leq\ & \|y_n\|^2 - 2\langle \omega, J(y_n)\rangle + \|\omega\|^2 + 2\rho f(\omega) \\
=\ & G(\omega, J(y_n)) \\
=\ & \|\omega\|^2 - 2\langle \omega, \beta_{n0}J(x_n) + \Sigma_{i=1}^{\infty}\beta_{ni}J(T_i x_n)\rangle \\
& + \|\beta_{n0}J(x_n) + \Sigma_{i=1}^{\infty}\beta_{ni}J(T_i x_n)\|^2 + 2\rho f(\omega) \\
\leq\ & \beta_{n0}G(\omega, J(x_n)) + \sum_{i=1}^{\infty}\beta_{ni}G(\omega, J(T_i x_n)) \\
\leq\ & G(\omega, J(x_n)),
\end{aligned}$$

i.e., $\omega \in Q_{k+1}$ and so $\cap_{i=1}^{\infty} F(T_i) \subset Q_n$ for all $n \geq 0$. Since f is proper convex and lower semicontinous, it follows from Lemma 20 that there exist $x^* \in E^*$ and $a \in R$ such that $f(y) \geq \langle y, x^*\rangle + a$ for all $y \in E$. Since $x_n = \Pi_{Q_n}^f x_0$ and $\omega \in Q_n$, from the definition of $\Pi_{Q_n}^f$, we have

$$\begin{aligned}
& G(\omega, J(x_0)) \\
\geq\ & G(x_n, J(x_0)) \\
=\ & \|x_n\|^2 - 2\langle x_n, J(x_0)\rangle + \|x_0\|^2 + 2\rho f(\omega) \\
\geq\ & \|x_0\|^2 - \|J(x_0) - \rho x^*\|^2 + 2\rho a.
\end{aligned}$$

Consequently, $\{x_n\}, \{y_n\}, \{T_i x_n\}$ for each $i \geq 1$ and $\{G(x_n, J(x_0))\}$ are bounded. Let

$$r = \sup_{n\geq 0}\{\|x_n\|, \|T_i x_n\| : i \geq 1\}.$$

Then it follows from Lemma 16 that there is a strictly increasing continuous and convex function $h : [0, 2r] \to R$ such that $h(0) = 0$ and, for any $j \geq 1$,

$$\begin{aligned}
& \|\beta_{n0}J(x_n) + \sum_{i=1}^{\infty}\beta_{ni}J(T_i x_n)\|^2 \\
\leq\ & \beta_{n0}\|x_n\|^2 + \sum_{i=1}^{\infty}\beta_{ni}\|T_i x_n\|^2 \\
& - \beta_{n0}\beta_{nj}h(\|J(x_n) - J(T_j x_n)\|).
\end{aligned}$$

By Lemma 15, for any $m \geq 1$, $x_{n+m} \in Q_{n+m} \subset Q_n$ and so

$$\phi(x_{n+m}, x_n) + G(x_n, J(x_0)) \leq G(x_{n+m}, J(x_0)).$$

By virtue of Remark 1, we have

$$G(x_n, J(x_0)) \leq G(x_{n+m}, J(x_0)),$$

which means that $\{G(x_n, J(x_0))\}$ is nondecreasing. Therefore, $\lim_{n\to\infty} G(x_n, J(x_0))$ exists and so

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0.$$

Particularly, $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Thus, by Lemma 14, we have

$$\begin{aligned}
& \lim_{n\to\infty} \|x_{n+m} - x_n\| \\
=\ & \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \quad \forall m \geq 1.
\end{aligned}$$

This yields that $\{x_n\}$ is a Cauchy sequence in C. Set $\lim_{n\to\infty} x_n = p \in C$. Since $x_{n+1} \in Q_{n+1} \subset Q_n$, it follows from the definition of Q_{n+1} that

$$G(x_{n+1}, J(y_n)) \leq G(x_{n+1}, J(x_n)).$$

Therefore, $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. This together with Lemma 14 yields that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{10}$$

In view of $\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$ and by (10), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|J(x_n) - J(y_n)\| = 0.$$

Let $\omega \in \cap_{i=1}^{\infty} F(T_i) \subset Q_{n+1}$ for all $n \geq 1$. Then, for each $j \geq 1$,

$$\begin{aligned}
& G(\omega, J(y_n)) \\
=\ & \|\omega\|^2 - 2\langle \omega, \beta_{n0}J(x_n) + \Sigma_{i=1}^{\infty}\beta_{ni}J(T_i x_n)\rangle \\
& + \|\beta_{n0}J(x_n) + \Sigma_{i=1}^{\infty}\beta_{ni}J(T_i x_n)\|^2 + 2\rho f(\omega) \\
\leq\ & G(\omega, J(x_n)) - \beta_{n0}\beta_{nj}h(\|J(x_n) - J(T_j x_n)\|).
\end{aligned}$$

Thus, for each $j \geq 1$,

$$\begin{aligned}
& \beta_{n0}\beta_{nj}h(\|J(x_n) - J(T_j x_n)\|) \\
\leq\ & G(\omega, J(x_n)) - G(\omega, J(y_n)) \\
=\ & 2\langle \omega, J(y_n) - J(x_n)\rangle + \|x_n\|^2 - \|y_n\|^2 \\
\leq\ & \|x_n - y_n\|(\|x_n\| + \|y_n\|) \\
& + 2\|\omega\|\|J(y_n) - J(x_n)\|.
\end{aligned}$$

From $\liminf_{n\to\infty}\beta_{n0}\beta_{nj}>0$ for each $j\geq 1$, it follows that

$$\lim_{n\to\infty} h(\|J(x_n)-J(T_jx_n)\|)=0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded subsets of E^*, by Lemma 21, we have

$$\lim_{n\to\infty}\|x_n-T_jx_n\|=0,\quad \forall j\geq 1.$$

Since T_j for each $j\geq 1$ is a closed mapping, we have $T_jp=p$ and so, $p\in\cap_{i=1}^\infty F(T_j)$.

Let $\bar{\omega}=\Pi^f_{\cap_{i=1}^\infty F(T_i)}x_0$. Since $x_n=\Pi^f_{Q_n}x_0$ and $\bar{\omega}\in\cap_{i=1}^\infty F(T_i)\subset Q_n$, one has

$$G(x_n,J(x_0))\leq G(\bar{\omega},J(x_0)).$$

By the weakly lower semicontinuity of norm and the lower semicontinuity of f, we obtain

$$\begin{aligned} G(p,J(x_0)) &\leq \liminf_{n\to\infty} G(x_n,J(x_0))\\ &\leq \limsup_{n\to\infty} G(x_n,J(x_0))\\ &\leq G(\bar{\omega},J(x_0)), \end{aligned}$$

i.e, $p=\bar{\omega}$. Therefore, the sequence $\{x_n\}$ converges strongly to a point $\Pi^f_{\cap_{i=1}^N F(T_i)}x_0$. This completes the proof. □

If $T_i: C\to C$ for each $i\geq 1$ is a relatively nonexpansive mapping in Theorem 26, we have the following:

Corollary 27 *Let C,E and f be the same as Theorem 26. Let $T_i: C\to C$ for each $i\geq 1$ be a relatively nonexpansive mapping such that $\cap_{i=1}^\infty F(T_i)\neq\emptyset$. Assume that $\{\beta_{ni}\}\subset[0,1]$ for each $i\geq 0$ such that $\liminf_{n\to\infty}\beta_{n0}\beta_{ni}>0$ for each $i\geq 1$ and $\sum_{j=0}^\infty\beta_{nj}=1$. Then the sequence $\{x_n\}$ generated by* **Algorithm I** *converges strongly to the point $\Pi^f_{\cap_{i=1}^\infty F(T_i)}x_0$.*

If $T_i=T$ for each $i\geq 1$ or, $f(x)=0$ for all $x\in E$ in Corollary 27, then, from Remarks 1 and 5, we can obtain the modified results of [7]:

Corollary 28 *[7] Let C,E and f be the same as Theorem 26, and let $T: C\to C$ be a relatively nonexpansive mapping such that $F(T)\neq\emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0\in C,\ Q_0=C,\\ y_n=J^{-1}(\beta_nJ(x_n)+(1-\beta_n)J(Tx_n)),\\ Q_{n+1}=\{z\in Q_n: G(z,J(y_n))\leq G(z,J(x_n))\},\\ x_{n+1}=\Pi^f_{Q_{n+1}}x_0\quad \forall n\geq 0, \end{cases}$$

where $\{\beta_n\}\subset[0,1]$ such that $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$. Then the sequence $\{x_n\}$ converges strongly to the point $\Pi^f_{F(T)}x_0$.

Corollary 29 *[7] Let C,E,T and f be the same as Corollary 28. Assume that $\{\beta_n\}\subset[0,1]$ such that $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0\in C,\ Q_0=C,\\ y_n=J^{-1}(\beta_nJ(x_n)+(1-\beta_n)J(Tx_n)),\\ Q_{n+1}=\{z\in Q_n:\phi(z,y_n)\leq\phi(z,x_n)\},\\ x_{n+1}=\Pi_{Q_{n+1}}x_0\quad \forall n\geq 0. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to the point $\Pi_{F(T)}x_0$.

Theorem 30 *Let C,E,f and $T_i(i\geq 1)$ be the same as Theorem 26 and let $\Omega_1\cap(\cap_{i=1}^\infty F(T_i))\neq\emptyset$. Suppose that the assumptions of Theorem 24 hold. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0\in C,\ Q_0=C,\\ y_n=J^{-1}(\beta_{n0}J(x_n)+\sum_{j=1}^\infty\beta_{nj}J(T_jx_n)),\\ u_n=U_{r_n}(y_n),\\ Q_{n+1}=\{z\in Q_n: G(z,J(u_n))\leq G(z,J(x_n))\},\\ x_{n+1}=\Pi^f_{Q_{n+1}}x_0,\quad \forall n\geq 0, \end{cases}$$

where U_r is defined in Theorem 24, $\{\beta_{ni}\}\subset[0,1]$ for each $i\geq 0$ and $\{r_n\}\subset(0,\infty)$ such that $\liminf_{n\to\infty}\beta_{n0}\beta_{nj}>0, \liminf_{n\to\infty}r_n>0$ and $\sum_{i=0}^\infty\beta_{ni}=1$ for all $j\geq 1, n\geq 1$. Then the sequence $\{x_n\}$ converges strongly to the point $\Pi^f_{\Omega_1\cap(\cap_{i=1}^\infty F(T_i))}x_0$.

Proof: By Lemmas 18 and 19, it follows that $\Omega_1\cap(\cap_{i=1}^\infty F(T_i))$ is closed and convex. Hence $\Pi^f_{\Omega_1\cap(\cap_{i=1}^\infty F(T_i))}$ is well-defined. As in the proof of Theorem 26, we can get that Q_n is nonempty closed and convex for all $n\geq 0$ and so $\{x_n\}$ is well-defined.

Let us, by induction, show that $\Omega_1\cap(\cap_{i=1}^\infty F(T_i))\subset Q_n$ for all $n\geq 0$. From $Q_0=C$, it follows that $\Omega_1\cap(\cap_{i=1}^\infty F(T_i))\subset Q_0$. Assume that $\Omega_1\cap(\cap_{i=1}^\infty F(T_i))\subset Q_k$ for some $k\geq 0$. Let $\omega\in\Omega_1\cap(\cap_{i=1}^\infty F(T_i))$. From Proposition 9 and Remark 11, it follows that

$$\begin{aligned} &2\rho f(\omega)\\ \leq\ & (\|\omega\|-\|u_k\|)^2+2\rho f(\omega)\\ \leq\ & \|\omega\|^2-2\langle\omega,J(u_k)\rangle+\|u_k\|^2+2\rho f(\omega)\\ =\ & G(\omega,J(u_k))\\ =\ & G(\omega,J(U_{r_k}y_k))\\ \leq\ & G(\omega,J(y_k))\\ =\ & \|\omega\|^2-2\langle\omega,J(y_k)\rangle+\|y_k\|^2+2\rho f(\omega)\\ \leq\ & \beta_{k0}G(\omega,J(x_k))+\sum_{j=1}^\infty\beta_{kj}G(\omega,J(T_jx_k))\\ \leq\ & G(\omega,J(x_k)), \end{aligned}$$

i.e., $\omega \in Q_{k+1}$. Therefore, $\Omega_1 \cap (\cap_{i=1}^{\infty} F(T_i)) \subset Q_n$ for all $n \geq 0$. Since $f : E \to R \cup \{+\infty\}$ is proper convex and lower semicontinuous, this, together with Lemma 20, yields that there exist $x^* \in E^*$ and $\alpha \in R$ such that $f(y) \geq \langle y, x^* \rangle + \alpha$ for all $y \in E$. Observe that

$$\begin{aligned} & G(x_n, J(x_0)) \\ = \; & \|x_n\|^2 - 2\langle x_n, J(x_0)\rangle + \|x_0\|^2 + 2\rho f(x_n) \\ \geq \; & (\|x_n\| - \|J(x_0) - \rho x^*\|)^2 + \|x_0\|^2 - \|J(x_0) \\ & - \rho x^*\|^2 + 2\rho\alpha. \end{aligned}$$

From both $x_n = \Pi_{Q_n}^f x_0$ and $\omega \in Q_n$, one concludes

$$\begin{aligned} & G(\omega, J(x_0)) \\ \geq \; & G(x_n, J(x_0)) \\ \geq \; & \|x_0\|^2 - \|J(x_0) - \rho x^*\|^2 + 2\rho\alpha. \end{aligned}$$

Thus $\{x_n\}$, $\{T_j x_n\}$ for each $j \geq 1$ and $\{G(x_n, J(x_0))\}$ are bounded. Let

$$r = \sup_{n\geq 0}\{\|x_n\|, \|T_j x_n\| : j \geq 1\}.$$

Then it follows from Lemma 16 that there exists a strictly increasing continuous and convex function $h : [0, 2r] \to R$ such that $h(0) = 0$ and, for each $j \geq 1$,

$$\begin{aligned} & \|\beta_{n0} J(x_n) + \sum_{j=1}^{\infty} \beta_{nj} J(T_j x_n)\|^2 \\ \leq \; & \beta_{n0}\|x_n\|^2 + \sum_{j=1}^{\infty} \beta_{nj}\|T_j x_n\|^2 \\ & - \beta_{n0}\beta_{nj} h(\|J(x_n) - J(T_j x_n)\|). \end{aligned}$$

Again, it follows from $x_{n+1} \in Q_{n+1} \subset Q_n$ and Lemma 15 that, for any $m \geq 1$,

$$\begin{aligned} & \phi(x_{n+m}, \Pi_{Q_n}^f x_0) + G(\Pi_{Q_n}^f x_0, J(x_0)) \\ \leq \; & G(x_{n+m}, J(x_0)), \end{aligned}$$

that is,

$$\phi(x_{n+m}, x_n) + G(x_n, J(x_0)) \leq G(x_{n+m}, J(x_0)).$$

Then, by Remark 1,

$$G(x_n, J(x_0)) \leq G(x_{n+m}, J(x_0))$$

and

$$G(x_n, J(x_0)) \leq G(x_{n+1}, J(x_0)).$$

Namely, $\{G(x_n, J(x_0))\}$ is nondecreasing. Thus $\lim_{n\to\infty} G(x_n, J(x_0))$ exists and so

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = \lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

By the definition of Q_n for each $n \geq 1$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n)$$

and so $\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$. From Lemma 14, it follows that

$$\begin{aligned} \lim_{n\to\infty} \|x_{n+1} - u_n\| &= \lim_{n\to\infty} \|x_{n+m} - x_n\| \\ &= \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \end{aligned}$$

This implies that $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $\{x_n\}$ is a Cauchy sequence in C. Set $\lim_{n\to\infty} x_n = \bar{x}$. Moreover, we have

$$\begin{aligned} & G(\omega, J(u_n)) \\ = \; & G(\omega, J(U_{r_n} y_n)) \\ \leq \; & G(\omega, J(y_n)) \\ \leq \; & \beta_{n0} G(\omega, J(x_n)) + \sum_{j=1}^{\infty} \beta_{nj} G(\omega, J(T_j x_n)) \\ & - \beta_{n0}\beta_{nj} h(\|J(x_n) - J(T_j x_n)\|), \\ \leq \; & G(\omega, J(x_n)) - \beta_{n0}\beta_{nj} h(\|J(x_n) - J(T_j x_n)\|). \end{aligned}$$

Consequently, for each $j \geq 1$,

$$\begin{aligned} & \beta_{n0}\beta_{nj} h(\|J(x_n) - J(T_j x_n)\|) \\ \leq \; & G(\omega, J(x_n)) - G(\omega, J(u_n)) \\ \leq \; & \|x_n - u_n\|(\|x_n\| + \|u_n\|) \\ & + 2\|\omega\|\|J(x_n) - J(u_n)\| \end{aligned}$$

and so $G(\omega, J(y_n)) \leq G(\omega, J(x_n))$. From $\liminf_{n\to\infty} \beta_{n0}\beta_{nj} > 0$, we derive that

$$\begin{aligned} & \lim_{n\to\infty} h(\|J(x_n) - J(Tx_n)\|) \\ = \; & \lim_{n\to\infty} G(\omega, J(x_n)) - G(\omega, J(u_n)) = 0. \end{aligned}$$

Furthermore, one has

$$\lim_{n\to\infty} \|J(x_n) - J(T_j x_n)\| = 0.$$

Since $J^* = J^{-1}$ is uniformly norm-to-norm continuous on bounded subsets of E^*, we get

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, \quad \forall j \geq 1. \tag{11}$$

Since T_j for each $j \geq 1$ is closed, it follows from (11) that $T_j\bar{x} = \bar{x}$ for each $j \geq 1$. Thus $\bar{x} \in \cap_{i=1}^{\infty} F(T_i)$. Noticing that $G(\omega, J(y_n)) \leq G(\omega, J(x_n))$. Then, by Proposition 9, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(U_{r_n} y_n, y_n) \\ &\leq G(\omega, J(y_n)) - G(\omega, J(U_{r_n} y_n)) \\ &\leq G(\omega, J(x_n)) - G(\omega, J(u_n)). \end{aligned}$$

Hence, $\lim_{n\to\infty} \phi(u_n, y_n) = 0$. By using Lemma 14, one has

$$\lim_{n\to\infty} \|u_n - y_n\| = 0.$$

This, together with $\lim_{n\to\infty} \|x_n - u_n\| = 0$, yields that $u_n \to \bar{x}$ and $y_n \to \bar{x}$. Since J is uniformly norm-to-norm continuous on bounded subsets of E,

$$\lim_{n\to\infty} \|J(u_n) - J(y_n)\| = 0.$$

In view of $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty} \frac{\|J(u_n) - J(y_n)\|}{r_n} = 0.$$

For the sake of brevity, let the function $H : C \times C \to R$ be the same as Theorem 24. Then

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, J(u_n) - J(y_n)\rangle \geq 0$$

for all $y \in C$. From the proof of Theorem 24, it follows that H satisfies the conditions $(C1)$-$(C4)$ of Assumption 2. This yields that

$$\begin{aligned} & \frac{1}{r_n}\langle y - u_n, J(u_n) - J(y_n)\rangle \\ \geq\ & -H(u_n, y) \\ \geq\ & H(y, u_n), \quad \forall y \in C. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$H(y, \bar{x}) \leq 0, \quad \forall y \in C.$$

Taking $y \in C$ arbitrarily, we have $\bar{x} \in C$ and $ty + (1-t)\bar{x} \in C$ for all $t \in (0, 1]$. Hence,

$$H(ty + (1-t)\bar{x}, \bar{x}) \leq 0$$

and so

$$\begin{aligned} 0 &= H(ty + (1-t)\bar{x}, ty + (1-t)\bar{x}) \\ &\leq tH(ty + (1-t)\bar{x}, y) \\ &\quad + (1-t)H(ty + (1-t)\bar{x}, \bar{x}) \\ &\leq tH(ty + (1-t)\bar{x}, y). \end{aligned}$$

Moreover, one has

$$H(ty + (1-t)\bar{x}, y) \geq 0$$

and so, from Assumption 2 $(C3)$,

$$0 \leq \limsup_{t\to 0+} H(ty + (1-t)\bar{x}, y) \leq H(\bar{x}, y).$$

This means $\bar{x} \in \Omega_1$ and so $\bar{x} \in \Omega_1 \cap (\cap_{i=1}^{\infty} F(T_i))$.

Next, let $\bar{\omega} = \Pi^f_{\Omega_1 \cap (\cap_{i=1}^{\infty} F(T_i))} x_0$. It follows from $x_{n+1} = \Pi^f_{Q_{n+1}} x_0$ and $\bar{\omega} \in \Omega_1 \cap (\cap_{i=1}^{\infty} F(T_i)) \subset Q_{n+1}$ that

$$G(x_{n+1}, J(x_0)) \leq G(\bar{\omega}, J(x_0)).$$

By the weakly lower semicontinuity of the norm, we have

$$\begin{aligned} & G(\bar{x}, J(x_0)) \\ =\ & \|\bar{x}\|^2 - 2\langle \bar{x}, J(x_0)\rangle + \|x_0\|^2 + 2\rho f(\bar{x}) \\ \leq\ & \liminf_{n\to\infty} G(x_n, J(x_0)) \\ \leq\ & \limsup_{n\to\infty} G(x_n, J(x_0)) \\ \leq\ & G(\bar{\omega}, J(x_0)). \end{aligned}$$

By Lemma 22, one has $\bar{x} = \bar{\omega}$. So the sequence $\{x_n\}$ converges strongly to the point $\Pi^f_{\Omega_1 \cap (\cap_{i=1}^{\infty} F(T_i))} x_0$. This completes the proof. □

If $f(x) = 0$ for all $x \in E$, then, from Remarks 1, 5 and Theorem 30, the following result holds:

Corollary 31 *Let C, E and $T_i (i \geq 1)$ be the same as Theorem 26, and let $\Omega_1 \cap (\cap_{i=1}^{\infty} F(T_i)) \neq \emptyset$. Suppose that the assumptions of Theorem 24 hold. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C,\ Q_0 = C, \\ y_n = J^{-1}(\beta_{n0} J(x_n) + \sum_{j=1}^{\infty} \beta_{nj} J(T_j x_n)), \\ u_n = U_{r_n}(y_n), \\ Q_{n+1} = \{z \in Q_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{Q_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where U_r is defined in Theorem 24, $\{\beta_{ni}\} \subset [0, 1]$ for each $i \geq 0$ and $\{r_n\} \subset (0, \infty)$ such that $\liminf_{n\to\infty} \beta_{n0}\beta_{nj} > 0, \liminf_{n\to\infty} r_n > 0$ and $\sum_{i=0}^{\infty} \beta_{ni} = 1$ for all $j \geq 1, n \geq 1$. Then the sequence $\{x_n\}$ converges strongly to the point $\Pi_{\Omega_1 \cap (\cap_{i=1}^{\infty} F(T_i))} x_0$.

If $A(x) = 0$, $\psi(x) = 0$ and $T_i = T$ for all $x \in C$ and $i \geq 1$ in Theorem 30 and Corollary 31, the following results hold:

Corollary 32 *Let C, E and f be the same as Theorem 26. Let $\Theta : C \times C \to R$ satisfy the conditions $(C1)$-$(C4)$ of Assumption 2, $T : C \to C$ be a closed and quasi-ϕ-nonexpansive mapping such that $EP(\Theta) \cap F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C,\ Q_0 = C, \\ y_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n) J(T x_n)), \\ u_n = U_{r_n} y_n, \\ Q_{n+1} = \{z \in Q_n : G(z, J(u_n)) \leq G(z, J(x_n))\}, \\ x_{n+1} = \Pi^f_{Q_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where

$$U_{r_n}y_n = \{z \in C : \Theta(z,y) + \frac{1}{r_n}\langle y-z, J(z) - J(y_n)\rangle \geq 0, \forall y \in C\},$$

$\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to the point* $\Pi^f_{F(T)\cap EP(\Theta)}x_0$.

Corollary 33 *Let* C, E, T *and* Θ *be the same as Corollary 32. Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{cases} x_0 \in C,\ Q_0 = C, \\ y_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(Tx_n)), \\ u_n = U_{r_n}y_n \\ Q_{n+1} = \{z \in Q_n : \phi(z, J(u_n)) \leq \phi(z, J(x_n))\}, \\ x_{n+1} = \Pi_{Q_{n+1}}x_0, \quad \forall n \geq 0, \end{cases}$$

where $U_{r_n}y_n$ *is the same as Corollary 32,* $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to the point* $\Pi_{F(T)\cap EP(\Theta)}x_0$.

Remark 34 *Corollary 33 improves Theorem TZ. Indeed, the iterative sequence of Corollary 33 is simpler than that of Theorem TZ in the following aspects:*

(1) $C_n = \{z \in C : \langle x_n - z, J(x_0) - J(x_n)\rangle \geq 0\}$ *is removed;*

(2) $x_{n+1} = \Pi_{C_n\cap Q_n}x_0$ *is replaced by* $x_{n+1} = \Pi_{Q_n}x_0$.

Remark 35 *From Theorem 23, we can derive some results for the problem (GMQEP2) which are similar to Theorems 30 and Corollaries 31-33.*

5 Applications

In this section, we shall utilize the obtained results in Section 4 to find a common element of the set of solutions of equilibrium problem (6) and the set of zeros of general H-monotone mapping in Banach spaces

Definition 36 *[7] A mapping* H *from* E *to* E^* *is said to be:*

(1) *monotone if*

$$\langle Hx - Hy, x - y\rangle \geq 0, \quad \forall x, y \in E;$$

(2) *strictly monotone if* H *is monotone and*

$$\langle Hx - Hy, x - y\rangle = 0 \Leftrightarrow x = y;$$

(3) β*-Lipschitz continuous if there exists a constant* $\beta \geq 0$ *such that*

$$\|Hx - Hy\| \leq \beta\|x - y\|, \forall x, y \in E.$$

Definition 37 *[7] Let* $M : E \to E^*$ *be a multivalued mapping with domain* $D(M) = \{z \in E : Mz \neq \emptyset\}$. *M is said to be:*

(1) *monotone if, for any* $x_i \in D(M)$ *and* $v_i \in Mx_i, i = 1, 2,$

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq 0;$$

(2) ι*-strongly monotone if, for any* $x_i \in D(M)$ *and* $v_i \in Mx_i, i = 1, 2,$

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq \lambda\|x_1 - x_2\|^2;$$

(3) *maximal monotone if* M *is monotone and its graph* $Gr(M) = \{(x, v) : v \in Mx\}$ *is not properly contained in the graph of any other monotone mapping;*

(4) general H*-monotone if* M *is monotone and* $(H + tM)E = E^*$ *holds for any* $t > 0$*, where* H *is a mapping from* E *to* E^*.

Throughout this section, without other specifications, let $H : E \to E^*$ be a strictly monotone mapping and $M : E \to 2^{E^*}$ be a general H-monotone mapping. From Li, Huang and O'Regan[7], we know that $T_\lambda = (H + \lambda M)^{-1}H : E \to D(M)$ is a single valued mapping and $F(T_\lambda) = M^{-1}(0)$ is nonempty closed and convex for all $\lambda > 0$.

The *modulus of convexity* of E is defined by

$$\delta_E(\epsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\Big\}.$$

The *modulus of smoothness* of E is defined by

$$\rho_E(t) = \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| \leq t\Big\}.$$

Lemma 38 *[7] Let* E *be a uniformly convex and uniformly smooth Banach space with* $\delta_E(\epsilon) \geq k\epsilon^2$ *and* $\rho_E(t) \leq ct^2$ *for some* $k, c > 0$. *Let* $H : E \to E^*$ *be a strictly monotone and* β*-Lipschtiz continuous mapping and* $M : E \to 2^{E^*}$ *be a general* H*-monotone and* ι*-strongly monotone mapping with* $\iota > 0$. *If there exists* $\lambda > 0$ *such that*

$$64c\beta^2 \leq \frac{1}{2}k\lambda^2\iota^2,$$

then T_λ *is a relatively nonexpansive mapping.*

Theorem 39 *Assume that the assumptions of Lemma 38 hold. Let* $\Theta : E \times E \to R$ *satisfy Assumption 2 with* $EP(\Theta) \cap M^{-1}(0) \neq \emptyset$, $f : E \to R \cup \{+\infty\}$

be a proper convex and lower semicontinuous mapping with bounded below and let $D(f) = E$. Define a sequence $\{x_n\}$ in E by the following algorithm:

$$\begin{cases} x_0 \in E,\ Q_0 = E, \\ y_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(T_\lambda x_n)), \\ u_n = U_{r_n}(y_n), \\ Q_{n+1} = \{z \in Q_n : G(z, J(u_n)) \le G(z, J(x_n))\}, \\ x_{n+1} = \Pi^f_{Q_{n+1}} x_0, \quad \forall n \ge 0, \end{cases}$$

where $T_{r_n} y_n = \{z \in E : \Theta(z,y) + \frac{1}{r_n}\langle y - z, J(z) - J(y_n)\rangle \ge 0, \forall y \in E\}$,$\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ converges strongly to the point $\Pi^f_{M^{-1}(0)\cap EP(\Theta)} x_0$.

Proof: It follows directly from Corollary 32 and Lemma 38, since $M^{-1}(0) = F(T_\lambda)$ is nonempty closed and convex for any $\lambda > 0$. □

Particularly, if $H = J$ is a normalized duality mapping from E to 2^{E^*}, the following results hold:

Theorem 40 *Let E, M, Θ and f be the same as Theorem 39. Let $J_\lambda = (J + \lambda M)^{-1} J$, where $\lambda > 0$. Let $\{x_n\}$ be a sequence in E defined as follows:*

$$\begin{cases} x_0 \in E,\ Q_0 = E, \\ y_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(J_\lambda x_n)), \\ u_n = T_{r_n} y_n, \\ Q_{n+1} = \{z \in Q_n : G(z, J(u_n)) \le G(z, J(x_n))\}, \\ x_{n+1} = \Pi^f_{Q_{n+1}} x_0, \quad \forall n \ge 0, \end{cases}$$

where $T_{r_n} y_n$ is defined in Theorem 39, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ converges strongly to the point $\Pi^f_{M^{-1}(0)\cap EP(\Theta)} x_0$.

Proof. It follows directly from Corollary 29 since J_λ is a relatively nonexpansive mapping (see [7]) and $M^{-1}(0) = F(J_\lambda)$ is nonempty closed and convex (see [18]).

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