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Semicontinuity for parametric Minty vector quasivariational inequalities in Hausdorff topological vector spaces

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Abstract This paper is devoted to the semicontinuity of solutions of a parametric generalized Minty vector quasivariational inequality problem with set-valued mappings [(in short (PGMVQVI)] in Hausdorff topological vector spaces, when the mapping and the constraint sets are perturbed by different parameters. The upper (lower) semicontinuity and closedness of the solution set mapping for (PGMVQVI) are established under some appropriate assumptions. The sufficient and necessary conditions of the Hausdorff lower semicontinuity and Hausdorff continuity of the solution set mapping for (PGMVQVI) are also derived without monotonicity. As an application, we discuss the upper semicontinuity for the solution set mapping of a special case of the (PGMVQVI).

Keywords Lower (upper) semicontinuity · Closedness · Hausdorff continuity · Nonlinear scalarization function · Gap function · Parametric generalized Minty vector quasivariational inequality

Mathematics Subject Classification 49J40 · 90C33

1 Introduction

Vector variational inequality problems [in short (VVI)] were firstly introduced by Giannessi (1980) in finite-dimensional spaces. Since then, extensive study of (VVI) has been done in

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finite and infinite spaces (see Giannessi 1998, 2000; Facchinei and Pang 2003; Chen et al. 2000, 2005a,b; Yang and Yao 2002; Chen and Wan 2011 and the references therein). (VVI) has been proved to be a very powerful tool of the current mathematical technology, which has been widely applied to transportation, finance and economics, mathematical physics, engineering sciences and so forth. In many situations, we always want to know the behavior of solution sets of VVI in practical problems when the problems' data vary. The stability of the solution set mappings for vector variational inequality and optimization with perturbed data is of great importance in variational inequalities and optimization theory. Several variants of stability, such as semicontinuity, continuity, Hölder continuity, Lipschitz continuity and some kinds of differentiability of the solution set mapping, have been studied for variational inequalities and equilibrium problems (see Zhong and Huang 2010; Li and Chen 2009; Barbagallo and Cojocaru 2009; Yen 1995; Huang et al. 2006; Chen et al. 2011, 2012a, 2013; Wong 2010 and the references therein).

Barbagallo and Cojocaru (2009) considered a class of scalar-type pseudo-monotone parametric variational inequalities in Banach space and showed that problems admitted continuous solutions with respect to the parameter. Khanh and Luu (2007) studied a class of Stampacchia type parametric multivalued quasivariational inequalities and obtained the semicontinuity of the solution sets and approximate solution sets. Recently, Zhong and Huang (2011a) studied the solution stability of parametric weak vector variational inequalities in reflexive Banach spaces. They obtained the lower semicontinuity of the solution mapping for the parametric weak vector variational inequalities with strictly C-pseudomapping, and also proved the lower semicontinuity of the solution mapping by degree-theoretic method. Aussel and Cotrina (2011) discussed the continuity properties of the strict and star solution mapping of a scalar quasivariational inequality in Banach spaces. Zhao (1997) established a sufficient and necessary condition (H1) for the Hausdorff lower semicontinuity of the solution mapping to parametric optimization problems. Under weaker assumptions, Kien (2005) also obtained the sufficient and necessary condition (H1) for the Hausdorff lower semicontinuity of the solution mapping to the problem of Zhao (1997). Using a condition (Hg) similar to that given in Zhao (1997), Li and Chen (2009) proved that (Hg) is also sufficient for the Hausdorff lower semicontinuity of the solution mapping to a class of weak vector variational inequality. Very recently, Chen et al. (2010) further extended the main results of Li and Chen (2009) to the parametric weak vector quasivariational inequality of Stampacchia type in Hausdorff topological vector spaces. Zhong and Huang (2011b) gave a key assumption $(\mathrm{Hg})'$ similar to (Hg) of Li and Chen (2009) and further proved a sufficient and necessary condition $(\mathrm{Hg})'$ for the Hausdorff lower semicontinuity and Hausdorff continuity of the solution mapping to parametric weak vector variational inequalities of Stampacchia type in reflexive Banach spaces.

On the other hand, Minty variational inequality has been shown to characterize some kinds of equilibrium more qualified than Stampacchia variational inequalities (see Giannessi 1998; Zhong and Huang 2010; Chen et al. 2012b and the references therein). Lalitha and Bhatia (2011) stated the importance of the solution stability for parametric quasivariational inequalities of the Minty type, presented various sufficient conditions for the upper and lower semicontinuity of solution sets as well as the approximate solution sets to a parametric scalar quasivariational inequality of the Minty type in finite-dimensional Euclidean spaces. To the best of our knowledge, these problems have received few attentions so far in infinite dimensions such as Banach spaces and Hausdorff topological vector spaces.

Inspired and motivated by the researches going on in this direction, we investigate the semicontinuity of solutions for a parametric generalized Minty vector quasivariational inequalities problem with set-valued mappings [in short (PGMVQVI)] in Hausdorff topological vector spaces, when the mapping and the constraint sets are perturbed by different parameters. Under some suitable assumptions, we establish the Berge and Hausdorff upper semicontinuity, Berge lower semicontinuity and closedness of the solution set mapping for (PGMVQVI). A parametric gap function is also introduced for (PGMVQVI) in Hausdorff topological vector spaces. By virtue of the parametric gap function, sufficient and necessary conditions of the Hausdorff lower semicontinuity and Hausdorff continuity of the solution set mapping for (PGMVQVI) are derived without monotonicity. As an application, we discuss the Berge and Hausdorff upper semicontinuity for a special case of (PGMVQVI). Moreover, examples are also provided for analyzing and illustrating the obtained results. The results presented in this paper develop, extend and improve some of the main results of Lalitha and Bhatia (2011).

2 Preliminaries

Throughout this paper, let M and $\bigwedge$ (the spaces of parameters) be two Hausdorff topological spaces, and X and Y be two locally convex Hausdorff topological vector spaces with the topological dual spaces X^* and Y^*, respectively. Let $L(X, Y)$ be the set of all linear continuous operators from X to Y, the value of a linear operator $t \in L(X, Y)$ at $x \in X$ is denoted by $\langle t, x\rangle$, and let $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a proper closed convex cone for all $x \in X$ with $\text{int}C(x) \neq \emptyset$. We always assume that $\langle \cdot, \cdot\rangle$ is continuous, 2^X denotes the family of all nonempty subsets of X.

We consider the following generalized Minty vector quasivariational inequality (GMVQVI): find $x \in K(x)$ such that

$$\langle t, y - x\rangle \notin -\text{int}C(x), \quad \forall y \in K(x),\ t \in T(y),$$

where $T : X \to 2^{L(X,Y)}$ and $K : X \to 2^X$ are two set-valued mappings.

If the mappings K and T are perturbed by parameters $\mu \in \bigwedge$ and $\lambda \in M$, respectively, then, for any given $(\mu, \lambda) \in \bigwedge \times M$, we define the PGMVQVI: find $x \in K(x, \mu)$ such that

$$\langle t, y - x\rangle \notin -\text{int}C(x), \quad \forall y \in K(x, \mu),\ t \in T(y, \lambda), \tag{1}$$

where $K : X \times \bigwedge \to 2^X$ and $T : X \times M \to 2^{L(X,Y)}$ are two set-valued mappings. If $\bigwedge = M$, $X = Y$, $\bigwedge$ is a nonempty closed subset of R^n and A is a nonempty closed and convex subset of Y, where $Y = R^m$, $\tilde{K} : Y \times \bigwedge \to 2^Y$ and $T : \bigwedge \times Y \to 2^Y$ are two set-valued mappings, $C(x) = R^m_+$ for all $x \in X$, set $K(x, \mu) = \tilde{K}(x, \mu) \cap A$, then (1) is reduced to the following parametric Minty quasivariational inequality corresponding to a parameter $\mu \in \bigwedge$: find $x \in \tilde{K}(x, \mu) \cap A$ such that

$$\langle t, x - y\rangle \leq 0, \quad \forall y \in \tilde{K}(x, \mu),\ t \in T(y, \mu), \tag{2}$$

which has been studied by Lalitha and Bhatia (2011).

For each $(\mu, \lambda) \in \bigwedge \times M$, let $E(\mu) = \{x \in X : x \in K(x, \mu)\}$, and denote the solutions set of (1) by $S(\mu, \lambda)$ corresponding to the parameters (μ, λ), i.e.,

$$S(\mu, \lambda) = \{x \in E(\mu) : \langle t, y - x\rangle \notin -\text{int}C(x), \quad \forall y \in K(x, \mu), t \in T(y, \lambda)\}.$$

We call the set-valued mapping $S : \bigwedge \times M \to 2^X$ as the solution mapping of (1). In the following, we always assume that $S(\mu, \lambda) \neq \emptyset$ for all $(\mu, \lambda) \in \bigwedge \times M$.

Our main concern is to study the semicontinuity of the solution mapping $S(\mu, \lambda)$, specifically, the Berge and Hausdorff lower semicontinuity, and Berge and Hausdorff upper semicontinuity of $S(\mu, \lambda)$ in Hausdorff topological vector spaces.

We first recall some definitions and basic results in the literature.

Definition 2.1 Chen et al. (2005a,b) The nonlinear scalarization function $\xi_e : X \times Y \to R$ is defined by

$$\xi_e(x, y) = \inf\{z \in R : y \in ze(x) - C(x)\}, \quad \forall (x, y) \in X \times Y,$$

where $e : X \to Y$ is vector-valued and $e(x) \in \text{int}C(x)$ for all $x \in X$.

Example 2.2 Chen et al. (2010) If $Y = R^n$, $e(x) = e$, and $C(x) = R^n_+$ for any $x \in X$, where $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in \text{int}R^n_+$, then the function $\xi_e(x, y) = \max_{1 \le i \le n}\{y_i\}$ is a nonlinear scalarization function for all $x \in X$, $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in Y$.

By Chen et al. (2005a, Theorem 2.1), Chen et al. (2010, Propositions 2.2 and 2.3), and Zhong and Huang (2011b, Lemma 2.3), the nonlinear scalarization function $\xi_e(\cdot, \cdot)$ has the following properties.

Proposition 2.3 *Let $e : X \to Y$ be a continuous selection from the set-valued map int$C(\cdot)$. For any given $x \in X$, $y \in Y$ and $r \in R$, the following results hold:*

(i) *If the mappings $C(\cdot)$ and $Y \setminus$ int$C(\cdot)$ are B-u.s.c on X, then $\xi_e(\cdot, \cdot)$ is continuous on $X \times Y$;*
(ii) *The mapping $\xi_e(x, \cdot) : Y \to R$ is convex;*
(iii) $\xi_e(x, y) < r \Leftrightarrow y \in re(x) - \text{int}C(x)$*;*
(iv) $\xi_e(x, y) \ge r \Leftrightarrow y \notin re(x) - \text{int}C(x)$*;*
(v) $\xi_e(x, re(x)) = r$*, especially,* $\xi_e(x, 0) = 0$.

Proposition 2.4 Chen et al. (2005a) *Let X and Y be two locally convex Hausdorff topological vector spaces; let $C : X \to 2^Y$ be a set-valued mapping such that, for each $x \in X$, $C(x)$ is a proper, closed, convex cone in Y with int$C(x) \ne \emptyset$, and let $e : X \to Y$ be a continuous selection from the set-valued map int$C(\cdot)$. Define a set-valued mapping $V : X \to 2^Y$ by $V(x) = Y \setminus$ int$C(x)$ for $x \in X$. Then, it holds that*

(i) *if $V(\cdot)$ is B-u.s.c on X, then $\xi_e(\cdot, \cdot)$ is upper semicontinuous on $X \times Y$;*
(ii) *if $C(\cdot)$ is B-u.s.c on X, then $\xi_e(\cdot, \cdot)$ is lower semicontinuous on $X \times Y$.*

Definition 2.5 Aubin and Ekeland (1984) Let Γ be a Hausdorff topological space, and X be a locally convex Hausdorff topological vector space. A set-valued mapping $F : \Gamma \to 2^X$ is said to be

(i) *Upper semicontinuous in the sense of Berge* (B-u.s.c) at $\gamma_0 \in \Gamma$ iff, for each open set V with $F(\gamma_0) \subset V$, there exists $\delta > 0$ such that

$$F(\gamma) \subset V, \quad \forall \gamma \in B(\gamma_0, \delta);$$

(ii) *Lower semicontinuous in the sense of Berge* (B-l.s.c) at $\gamma_0 \in \Gamma$ iff, for each open set V with $F(\gamma_0) \cap V \ne \emptyset$, there exists $\delta > 0$ such that

$$F(\gamma) \cap V \ne \emptyset, \quad \forall \gamma \in B(\gamma_0, \delta);$$

(iii) *Upper semicontinuous in the sense of Hausdorff* (H-u.s.c) at $\gamma_0 \in \Gamma$ iff, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(\gamma) \subset U(F(\gamma_0), \epsilon), \quad \forall \gamma \in B(\gamma_0, \delta);$$

(iv) *Lower semicontinuous in the sense of Hausdorff* (H-l.s.c) at $\gamma_0 \in \Gamma$ iff, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(\gamma_0) \subset U(F(\gamma), \epsilon), \quad \forall \gamma \in B(\gamma_0, \delta);$$

(v) *Closed* iff, the graph of F is closed, i.e., the set $G(F) = \{(\gamma, x) \in \Gamma \times X : x \in F(\gamma)\}$ is closed in $\Gamma \times X$.

In $B(\gamma_0, \delta)$, γ_0 is the center and δ is the radius of ball in Γ. We say F is H-l.s.c (resp. H-u.s.c, B-l.s.c, B-u.s.c) on Γ iff it is H-l.s.c (resp. H-u.s.c, B-l.s.c, B-u.s.c) at each $\gamma \in \Gamma$. F is called continuous (resp. H-continuous) on Γ iff it is both B-l.s.c (resp. H-l.s.c) and B-u.s.c (resp. H-u.s.c) on Γ.

Definition 2.6 Berge (1963) A set $B \subset X$ is said to be *balanced* if $\rho B \subset B$ for each $\rho \in R$ with $|\rho| \leq 1$.

Lemma 2.7 Aubin and Ekeland (1984)

(i) *F is B-l.s.c at $\gamma_0 \in \Gamma$ if and only if, for any net $\{\gamma_\alpha\} \subseteq \Gamma$ with $\gamma_\alpha \to \gamma_0$ and $x_0 \in F(\gamma_0)$, there exists a net $\{x_\alpha\} \subseteq X$ with $x_\alpha \in F(\gamma_\alpha)$ for all α, such that $x_\alpha \to x_0$.*
(ii) *If F is compact-valued, then F is B-u.s.c at $\gamma_0 \in \Gamma$ if and only if, for any net $\{\gamma_\alpha\} \subseteq \Gamma$ with $\gamma_\alpha \to \gamma_0$ and for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \in F(\gamma_\alpha)$ for all α, there exist $x_0 \in F(\gamma_0)$ and a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x_\beta \to x_0$.*
(iii) *If F is B-u.s.c and closed-valued, then F is closed. Conversely, if F is closed and X is compact, then F is B-u.s.c.*

Lemma 2.8 Bank et al. (1982) *Let Γ be a Hausdorff topological space, and X a locally convex Hausdorff topological vector space, and let $F : \Gamma \to 2^X$ be a set-valued mapping and $\gamma_0 \in \Gamma$ be a given point.*

(i) *If F is B-u.s.c at γ_0, then F is H-u.s.c at γ_0. Conversely, if F is H-u.s.c at γ_0 and $F(\gamma_0)$ is compact, then F is B-u.s.c at γ_0.*
(ii) *If F is H-l.s.c at γ_0, then F is B-l.s.c at γ_0. Conversely, if F is B-l.s.c at γ_0 and $cl(F(\gamma_0))$ is compact, then F is H-l.s.c at γ_0.*

Lemma 2.9 Berge (1963) *For each neighborhood U of 0_X, there exists a balanced open neighborhood $\bigcirc$ of 0_X such that $\bigcirc + \bigcirc + \bigcirc \subset U$.*

3 Upper semicontinuity

In this section, we shall study the Berge (Hausdorff) upper semicontinuity of the solution mapping $S(\mu, \lambda)$ for the (PGMVQVI) corresponding to a pair of parameters (μ, λ).

Theorem 3.1 *Assume that the following conditions are satisfied:*

(i) *$E(\cdot)$ is B-u.s.c with compact values on $\bigwedge$;*
(ii) *$K(\cdot, \cdot)$ is B-l.s.c on $X \times \bigwedge$;*
(iii) *$T(\cdot, \cdot)$ is B-l.s.c with compact values on $X \times M$;*
(iv) *$W(\cdot) = Y \setminus -intC(\cdot)$ is closed on X. Then the solution mapping $S(\cdot, \cdot)$ is H-u.s.c and closed on $\bigwedge \times M$.*

Proof Suppose to the contrary that there exists $(\mu_0, \lambda_0) \in \bigwedge \times M$ such that $S(\cdot, \cdot)$ is not B-u.s.c at (μ_0, λ_0). Then there exists an open set V satisfying $S(\mu_0, \lambda_0) \subset V$, and net $\{(\mu_\alpha, \lambda_\alpha)\}$ and $x_\alpha \in S(\mu_\alpha, \lambda_\alpha)$ such that $(\mu_\alpha, \lambda_\alpha) \to (\mu_0, \lambda_0)$ and $x_\alpha \notin V$ for all α. Since $x_\alpha \in S(\mu_\alpha, \lambda_\alpha)$, then $x_\alpha \in E(\mu_\alpha)$. By assumption (i), $E(\cdot)$ is B-u.s.c with compact values at μ_0. Then there exists $x_0 \in E(\mu_0)$ such that $x_\alpha \to x_0$ (here we may take a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ if necessary).

Suppose that $x_0 \notin S(\mu_0, \lambda_0)$, i.e., there exists $y_1 \in K(x_0, \mu_0)$, for each $t_1 \in T(y_1, \lambda_0)$ or, for each $\bar{y} \in K(x_0, \mu_0)$, there exists $\bar{t} \in T(\bar{y}, \lambda_0)$ such that

$$\langle t_1, y_1 - x_0 \rangle \in -\text{int}C(x_0) \tag{3}$$

or,

$$\langle \bar{t}, \bar{y} - x_0 \rangle \in -\text{int}C(x_0). \tag{4}$$

Taking into account $x_\alpha \in S(\mu_\alpha, \lambda_\alpha)$ that

$$\langle t_\alpha, z_\alpha - x_\alpha \rangle \notin -\text{int}C(x_\alpha), \quad \forall z_\alpha \in K(x_\alpha, \mu_\alpha),\ t_\alpha \in T(z_\alpha, \lambda_\alpha).$$

Since $K(\cdot, \cdot)$ is B-l.s.c at (x_0, μ_0), for any net $\{(x_\alpha, \mu_\alpha)\} \subseteq X \times \bigwedge$ with $(x_\alpha, \mu_\alpha) \to (x_0, \mu_0)$ and any $z_0 \in K(x_0, \mu_0)$, there exists $z_\alpha \in K(x_\alpha, \mu_\alpha)$ such that $z_\alpha \to z_0$. Again from (iii), $T(\cdot, \cdot)$ is B-l.s.c with compact values at (z_0, λ_0). For any net $\{(z_\alpha, \lambda_\alpha)\} \subseteq X \times M$ with $(z_\alpha, \lambda_\alpha) \to (z_0, \lambda_0)$ and any $t_0 \in T(z_0, \lambda_0)$, there exists $t_\alpha \in T(z_\alpha, \lambda_\alpha)$ such that $t_\alpha \to t_0$. Therefore, $\langle t_\alpha, z_\alpha - x_\alpha \rangle \to \langle t_0, z_0 - x_0 \rangle$. By (iv), one has

$$\langle t_0, z_0 - x_0 \rangle \in Y \setminus -\text{int}C(x_0),$$

that is,

$$\langle t_0, z_0 - x_0 \rangle \notin -\text{int}C(x_0), \quad \forall z_0 \in K(x_0, \mu_0),\ t_0 \in T(z_0, \lambda_0),$$

which contradicts (3) and (4). This implies that $x_0 \in S(\mu_0, \lambda_0) \subset V$, which leads to a contradiction. Since $x_\alpha \notin V, \forall \alpha, x_\alpha \to x_0$ and V is open. Thus, $S(\cdot, \cdot)$ is B-u.s.c at each $(\mu_0, \lambda_0) \in \bigwedge \times M$. By Lemma 2.8, we know that $S(\cdot, \cdot)$ is H-u.s.c at each $(\mu_0, \lambda_0) \in \bigwedge \times M$.

On the other hand, we show that $S(\cdot, \cdot)$ is closed at each $(\mu_0, \lambda_0) \in \bigwedge \times M$. Taking $x_\alpha \in S(\mu_\alpha, \lambda_\alpha)$ with $(\mu_\alpha, \lambda_\alpha) \to (\mu_0, \lambda_0)$ and $x_\alpha \to x_0$. Then, $x_\alpha \in E(\mu_\alpha)$. Together with (i), this yields that $x_0 \in E(\mu_0)$. By the same proof as above, we have $x_0 \in S(\mu_0, \lambda_0)$, which implies that $S(\cdot, \cdot)$ is closed on $\bigwedge \times M$. □

Remark 3.2 From the proof of Theorem 3.1, we know that, if all conditions of Theorem 3.1 are satisfied, then the solution mapping $S(\cdot, \cdot)$ is B-u.s.c on $\bigwedge \times M$.

Remark 3.3 Chen et al. (2010) Since the set-valued mapping $E : \bigwedge \to 2^X$ is related with the mapping $K : X \times \bigwedge \to 2^X$, if K is B-u.s.c with closed values and X is a compact space, then the mapping E is B-u.s.c.

From Theorem 3.1, we can conclude the following corollary:

Corollary 3.4 *Let $(\mu_0, \lambda_0) \in \bigwedge \times M$ be a given point. If $E(\mu_0)$ is a compact set, $K(\cdot, \cdot)$ is B-l.s.c on $X \times \{\mu_0\}$, $T(\cdot, \cdot)$ is B-l.s.c with compact values on $X \times \{\lambda_0\}$, and $W(\cdot) = Y \setminus -\text{int}C(\cdot)$ is closed on X, then $S(\mu_0, \lambda_0)$ is a compact set.*

Theorem 3.5 *Let $(\mu_0, \lambda_0) \in \bigwedge \times M$ be a given point. Assume that the following conditions are satisfied:*

(i) *$E(\cdot)$ is B-u.s.c with compact values on $\bigwedge$.*
(ii) *For any $x_0 \in K(x_0, \mu_0)$, $(x_\alpha, \mu_\alpha, \lambda_\alpha) \to (x_0, \mu_0, \lambda_0)$ and*

$$\langle t_0, y_0 - x_0 \rangle \in -intC(x_0) \tag{5}$$

for some $y_0 \in K(x_0, \mu_0)$ and $t_0 \in T(y_0, \lambda_0)$ implies that there exists α such that

$$\langle t_\alpha, y_\alpha - x_\alpha \rangle \in -intC(x_\alpha) \tag{6}$$

for some $y_\alpha \in K(x_\alpha, \mu_\alpha)$ and $t_\alpha \in T(y_\alpha, \lambda_\alpha)$. Then the solution mapping $S(\cdot, \cdot)$ is H-u.s.c at (μ_0, λ_0).

Proof Suppose to the contrary that the solution mapping $S(\cdot, \cdot)$ is not B-u.s.c at (μ_0, λ_0). Then there exists an open set V such that $S(\mu_0, \lambda_0) \subset V$, and net $\{(\mu_\alpha, \lambda_\alpha)\} \subseteq \bigwedge \times M$ and $x_\alpha \in S(\mu_\alpha, \lambda_\alpha)$ such that $(\mu_\alpha, \lambda_\alpha) \to (\mu_0, \lambda_0)$ and $x_\alpha \notin V$, $\forall \alpha$. Since $x_\alpha \in E(\mu_\alpha)$ and $E(\cdot)$ is B-u.s.c with compact values at μ_0, there is an $x_0 \in E(\mu_0)$ such that $x_\alpha \to x_0 \in E(\mu_0) \setminus V$. Hence, $x_0 \in K(x_0, \mu_0)$ and $(x_\alpha, \mu_\alpha, \lambda_\alpha) \to (x_0, \mu_0, \lambda_0)$. By virtue of $x_\alpha \in S(\mu_\alpha, \lambda_\alpha)$, we get

$$\langle t_\alpha, y_\alpha - x_\alpha \rangle \notin -\text{int}C(x_\alpha), \quad \forall y_\alpha \in K(x_\alpha, \mu_\alpha), t_\alpha \in T(y_\alpha, \lambda_\alpha).$$

By (ii), we have

$$\langle t_0, y_0 - x_0 \rangle \notin -\text{int}C(x_0), \quad \forall y_0 \in K(x_0, \mu_0), \ t_0 \in T(y_0, \lambda_0),$$

i.e., $x_0 \in S(\mu_0, \lambda_0)$, which contradicts that $x_0 \in E(\mu_0) \setminus V$. Thus, the solution mapping $S(\cdot, \cdot)$ is B-u.s.c at (μ_0, λ_0). By Lemma 2.8, $S(\cdot, \cdot)$ is H-u.s.c at (μ_0, λ_0). □

Remark 3.6 From the proof of Theorem 3.5, we know that, if the conditions (i) and (ii) of Theorem 3.5 hold, then the solution mapping $S(\cdot, \cdot)$ is also B-u.s.c on $\bigwedge \times M$.

Remark 3.7 It is easy to see that the conditions (ii)–(iv) of Theorem 3.1 implies the condition (ii) of Theorem 3.5. Indeed, the condition (ii) of Theorem 3.5 is a very mild condition, which means that, if, for some $y_0 \in K(x_0, \mu_0)$, $t_0 \in T(y_0, \lambda_0)$, the inequality (5) is satisfied, then the similar inequality (6) must be preserved for some $y_\alpha \in K(x_\alpha, \mu_\alpha), t_\alpha \in T(y_\alpha, \lambda_\alpha)$ corresponding to some net $(x_\alpha, \mu_\alpha, \lambda_\alpha)$ with $(x_\alpha, \mu_\alpha, \lambda_\alpha) \to (x_0, \mu_0, \lambda_0)$. However, if $S(\cdot, \cdot)$ is B-u.s.c and H-u.s.c at (μ_0, λ_0), the condition (ii) of Theorem 3.5 does not imply the conditions (ii)–(iv) of Theorem 3.1.

Example 3.8 Let $\bigwedge = M = (-1, 1)$, $X = Y = R$, and $C(x) = R_+$ for all $x \in X$. Define the set-valued mappings $K : X \times \bigwedge \to 2^X$ and $T : X \times M \to 2^Y$ by

$$T(x, \lambda) = \begin{cases} [1, 2], & \text{if } x = 0, \\ \{3\}, & \text{if } x \neq 0, \end{cases}$$

and

$$K(x, \mu) = \begin{cases} \{0, 1\}, & \text{if } x = 0, \\ \{\chi \in X : \chi \in [0, |\mu|]\}, & \text{if } x \neq 0. \end{cases}$$

Then, for every $\mu \in \bigwedge$, $E(\mu) = [0, |\mu|] \cup \{1\}$. So, $E(\cdot)$ is B-u.s.c on $\bigwedge$. For $(\mu_0, \lambda_0) = (0, 0)$, Theorem 3.5(ii) is satisfied. After computation, $S(\mu, \lambda) = \{0\}$ for all $(\mu, \lambda) \in \bigwedge \times M$. Thus, $S(\cdot, \cdot)$ is B-u.s.c at (μ_0, λ_0). But the mappings K and T are not B-u.s.c at $(0, 0)$.

4 Lower semicontinuity

In this section, we study the Berge and Hausdorff lower semicontinuity of the solution mapping $S(\mu,\lambda)$ for the (PGMVQVI) corresponding to a pair of parameters (μ,λ).

Theorem 4.1 *For any given $(\mu_0,\lambda_0)\in\bigwedge\times M$, if the following conditions are satisfied:*

(i) *$E(\cdot)$ is B-l.s.c on $\bigwedge$.*

(ii) *For any $x_0\in K(x_0,\mu_0)$, $(x_\alpha,\mu_\alpha,\lambda_\alpha)\to(x_0,\mu_0,\lambda_0)$ and*

$$\langle t, y-x_0\rangle\notin -intC(x_0),\quad \forall y\in K(x_0,\mu_0),\ t\in T(y,\lambda_0) \tag{7}$$

implies that there exists α such that

$$\langle t_\alpha, y_\alpha-x_\alpha\rangle\notin -intC(x_\alpha),\quad \forall y_\alpha\in K(x_\alpha,\mu_\alpha),\ t_\alpha\in T(y_\alpha,\lambda_\alpha). \tag{8}$$

Then the solution mapping $S(\cdot,\cdot)$ is B-l.s.c at (μ_0,λ_0).

Proof Suppose to the contrary that the solution mapping $S(\cdot,\cdot)$ is not B-l.s.c at (μ_0,λ_0). Then there exist $(\mu_\alpha,\lambda_\alpha)\in\bigwedge\times M$ with $(\mu_\alpha,\lambda_\alpha)\to(\mu_0,\lambda_0)$ and $x_0\in E(\mu_0)$ such that for every sequence $x_\alpha\in S(\mu_\alpha,\lambda_\alpha)$, $x_\alpha\not\to x_0$. Since $E(\cdot)$ is B-l.s.c at μ_0, $\mu_\alpha\to\mu_0$ and $x_0\in E(\mu_0)$, there exists $\tilde{x}_\alpha\in E(\mu_\alpha)$ such that $\tilde{x}_\alpha\to x_0$.

Without loss of generality, taking into account the contradiction assumption, set $\tilde{x}_\alpha\notin S(\mu_\alpha,\lambda_\alpha)$. Then, for some $\tilde{y}_\alpha\in K(\tilde{x}_\alpha,\mu_\alpha)$ and $\tilde{t}_\alpha\in T(\tilde{y}_\alpha,\lambda_\alpha)$,

$$\langle\tilde{t}_\alpha,\tilde{y}_\alpha-\tilde{x}_\alpha\rangle\in-\text{int}C(\tilde{x}_\alpha). \tag{9}$$

Due to $x_0\in S(\mu_0,\lambda_0)$, it implies that

$$\langle t_0, y_0-x_0\rangle\notin-\text{int}C(x_0),\quad\forall y_0\in K(x_0,\mu_0),\ t_0\in T(y_0,\lambda_0). \tag{10}$$

In view of $(\tilde{x}_\alpha,\mu_\alpha,\lambda_\alpha)\to(x_0,\mu_0,\lambda_0)$ and from (ii), there exists α such that

$$\langle t_\alpha, y_\alpha-\tilde{x}_\alpha\rangle\notin-\text{int}C(\tilde{x}_\alpha),\quad\forall y_\alpha\in K(\tilde{x}_\alpha,\mu_\alpha),\ t_\alpha\in T(y_\alpha,\lambda_\alpha),$$

which contradicts (9). □

Remark 4.2 The condition (ii) of Theorem 4.1 is a very weak condition, which can be explained that, if the inequality (7) is satisfied for all $y\in K(x_0,\mu_0)$, $t\in T(y,\lambda_0)$, then the similar inequality (8) must be preserved for all $y_\alpha\in K(x_\alpha,\mu_\alpha)$, $t_\alpha\in T(y_\alpha,\lambda_\alpha)$ corresponding to some net $\{(x_\alpha,\mu_\alpha,\lambda_\alpha)\}$ with $(x_\alpha,\mu_\alpha,\lambda_\alpha)\to(x_0,\mu_0,\lambda_0)$.

Now we give an example to show that the condition (ii) in Theorem 4.1 is indispensable.

Example 4.3 Let $\bigwedge=M=(-1,1)$, $X=R$, $Y=R^2$, $(\mu_0,\lambda_0)=(0,0)\in\bigwedge\times M$ and $C(x)=R^2_+$ for all $x\in X$. Define the set-valued mappings $K:X\times\bigwedge\to 2^X$ and $T:X\times M\to 2^Y$ by

$$T(x,\lambda)=\begin{cases}\{(1,\ell)^{\mathrm{T}}\in Y:\ell\in[-|\lambda|,|\lambda|+1]\}, & \text{if } x<1,\\ \{(1,1+\lambda^2)^{\mathrm{T}}\in Y\}, & \text{if } x\geq 1,\end{cases}$$

and

$$K(x,\mu)=\begin{cases}\{\chi\in X:\chi\in[1,2x-1]\}, & \text{if } x\geq 1,\\ \{\chi\in X:\chi\in[x,1+|\mu|]\}, & \text{if } 0\leq x<1,\\ \{0\}, & \text{if } x<0.\end{cases}$$

After computation, $S(\mu_0, \lambda_0) = (-\infty, 0] \cup \{1\}$ and $S(\mu, \lambda) = \{1\}$ for all $(\mu, \lambda) \in \bigwedge \times M$ and $(\mu, \lambda) \neq (\mu_0, \lambda_0)$, $E(\mu) = R$ for all $\mu \in \bigwedge$. Therefore, $E(\cdot)$ is B-l.s.c on $\bigwedge$ and $S(\cdot, \cdot)$ is not B-l.s.c at (μ_0, λ_0). It is easy to check that Theorem 4.1(ii) is not satisfied. In fact, taking $(x_n, \mu_n, \lambda_n) = (1 + \frac{1}{n}, \frac{1}{n}, \frac{1}{n})$, $n \in N$. Then, $(x_n, \mu_n, \lambda_n) \to (1, 0, 0)$ as $n \to \infty$ and

$$\langle t, y - x_0 \rangle \notin -\text{int}R^2_+, \quad \forall y \in K(1, 0),\ t \in T(y, 0).$$

However, there exist $y_n \in K(x_n, \mu_n)$ and $t_n = (1, 1 + \frac{1}{n^2}) \in T(1, \lambda_n)$ such that

$$\langle t_n, y_n - x_n \rangle = \left(-\frac{1}{n}, -\frac{1}{n}\left(1 + \frac{1}{n^2}\right)\right)^{\mathrm{T}} \in -\text{int}R^2_+, \quad \forall n \in N.$$

If the conditions in Theorem 4.1 are strengthened, then we can get the Hausdorff lower semicontinuity of the solution mapping $S(\cdot, \cdot)$.

Theorem 4.4 *For any given $(\mu_0, \lambda_0) \in \bigwedge \times M$, if the following conditions are satisfied:*

(i) *$E(\cdot)$ is B-l.s.c with compact values on $\bigwedge$.*

(ii) *For any $x_0 \in K(x_0, \mu_0)$, $(x_\alpha, \mu_\alpha, \lambda_\alpha) \to (x_0, \mu_0, \lambda_0)$ and*

$$\langle t, y - x_0 \rangle \notin -intC(x_0), \quad \forall y \in K(x_0, \mu_0),\ t \in T(y, \lambda_0)$$

implies that there exists α such that

$$\langle t_\alpha, y_\alpha - x_\alpha \rangle \notin -intC(x_\alpha), \quad \forall y_\alpha \in K(x_\alpha, \mu_\alpha),\ t_\alpha \in T(y_\alpha, \lambda_\alpha).$$

Then the solution mapping $S(\cdot, \cdot)$ is H-l.s.c at (μ_0, λ_0).

Proof By Theorem 4.1, we know that the solution mapping $S(\cdot, \cdot)$ is B-l.s.c at (μ_0, λ_0). Taking into account Lemma 2.7 and condition (i) that $S(\mu_0, \lambda_0)$ is compact. It follows from Lemma 2.8 that $S(\cdot, \cdot)$ is H-l.s.c at (μ_0, λ_0). □

By the nonlinear scalarization function ξ_e, we introduce the following so-called "gap" function. Suppose that $K(x, \mu)$ is a compact set for each $(x, \mu) \in X \times \bigwedge$, $T(x, \lambda)$ is also a compact set for each $(x, \lambda) \in X \times M$, $V(\cdot) =: Y \setminus \text{int}C(\cdot)$ and $C(\cdot)$ are B-u.s.c on X. Define a function $g : \bigwedge \times M \times X \to R$ by

$$g(\mu, \lambda, x) =: \min_{t \in T(y,\lambda), y \in K(x,\mu)} \xi_e(x, \langle t, y - x \rangle), \quad \forall x \in E(\mu). \tag{11}$$

Since $K(x, \mu)$ and $T(x, \lambda)$ are compact sets and $\xi_e(\cdot, \cdot)$ is continuous, then $g(\mu, \lambda, x)$ is well defined. We shall use the function $g(\mu, \lambda, x)$ to establish the lower semicontinuity of the solution mapping of (PGMVQVI). Firstly, we study the properties of the function $g(\cdot, \cdot, \cdot)$, and relations between $g(\cdot, \cdot, \cdot)$ and the solution mapping $S(\cdot, \cdot)$.

Lemma 4.5 (i) $g(\mu_0, \lambda_0, x_0) = 0$ *if and only if* $x_0 \in S(\mu_0, \lambda_0)$*;*

(ii) $g(\mu, \lambda, x) < 0$ *for all* $x \in E(\mu) \setminus S(\mu, \lambda)$*;*

(iii) $g(\mu, \lambda, x) \leq 0$ *for all* $x \in E(\mu)$.

Proof (i) If $g(\mu_0, \lambda_0, x_0) = 0$, then

$$g(\mu_0, \lambda_0, x_0) = \min_{t \in T(y,\lambda_0), y \in K(x_0,\mu_0)} \xi_e(x_0, \langle t, y - x_0 \rangle) = 0,$$

this shows that

$$\xi_e(x_0, \langle t, y - x_0\rangle) \geq 0, \quad \forall y \in K(x_0, \mu_0),\ t \in T(y, \lambda_0).$$

By Proposition 2.3, one has

$$\langle t, y - x_0\rangle \notin -\mathrm{int}C(x_0), \quad \forall y \in K(x_0, \mu_0),\ t \in T(y, \lambda_0).$$

Therefore, $x_0 \in S(\mu_0, \lambda_0)$.
Conversely, if $x_0 \in S(\mu_0, \lambda_0)$, then

$$\langle t, y - x_0\rangle \notin -\mathrm{int}C(x_0), \quad \forall y \in K(x_0, \mu_0),\ t \in T(y, \lambda_0).$$

From Proposition 2.3, it follows that

$$\xi_e(x_0, \langle t, y - x_0\rangle) \geq 0, \quad \forall y \in K(x_0, \mu_0),\ t \in T(y, \lambda_0).$$

Moreover, we have

$$g(\mu_0, \lambda_0, x_0) = \min_{t\in T(y,\lambda_0), y\in K(x_0,\mu_0)} \xi_e(x_0, \langle t, y - x_0\rangle) \geq 0. \tag{12}$$

Again from Proposition 2.3, one has

$$g(\mu_0, \lambda_0, x_0) \leq \min_{t\in T(x_0,\lambda_0)} \xi_e(x_0, \langle t, x_0 - x_0\rangle) = 0. \tag{13}$$

As a consequence, from (12) and (13), we have $g(\mu_0, \lambda_0, x_0) = 0$.

(ii) For any given $x \in E(\mu)$, but $x \notin S(\mu, \lambda)$, there exists $\bar{y}_1 \in K(x, \mu)$ for all $\bar{t} \in T(\bar{y}_1, \lambda)$ or, for all $y \in K(x, \mu)$, there exists $t_0 \in T(y, \lambda)$ such that

$$\langle \bar{t}, \bar{y}_1 - x\rangle \in -\mathrm{int}C(x) \tag{14}$$

or,

$$\langle t_0, y - x\rangle \in -\mathrm{int}C(x). \tag{15}$$

Taking into account Proposition 2.3 and from (14) and (15) that

$$\xi_e(x, \langle \bar{t}, \bar{y}_1 - x\rangle) < 0, \quad \xi_e(x, \langle t_0, y - x\rangle) < 0.$$

Then, for any $x \in E(\mu) \setminus S(\mu, \lambda)$,

$$g(\mu, \lambda, x) = \min_{t\in T(y,\lambda), y\in K(x,\mu)} \xi_e(x, \langle t, y - x\rangle) < 0.$$

(iii) It directly follows from (i) and (ii). □

Remark 4.6 If (i) and (iii) of Lemma 4.5 hold, then the function g is called a parametric gap function for (PGMVQVI). Gap function is an important method for solving variational inequalities and widely applied in optimization problems, equation problems and so on (see Chen et al. 2000; Yang and Yao 2002; Li et al. 2006; Noor 2006 and the references therein).

Lemma 4.7 *Let $E(\mu)$ be nonempty for each $\mu \in \bigwedge$. Assume that the following conditions are satisfied:*

(i) *$K(\cdot, \cdot)$ is B-u.s.c with compact values on $X \times \bigwedge$;*
(ii) *$T(\cdot, \cdot)$ is B-u.s.c with compact values on $X \times M$;*
(iii) *$C(\cdot)$ is B-u.s.c on X, and $e(\cdot) \in intC(\cdot)$ is continuous on X.*

Then, $g(\cdot, \cdot, \cdot)$ is a lower semicontinuous function.

Proof By the assumptions and (11), $g(\mu, \lambda, x)$ is finite for any $\mu \in \bigwedge, \lambda \in M$ and $x \in E(\mu)$. Set $l \in R$. Assume that net $\{(\mu_\alpha, \lambda_\alpha, x_\alpha)\} \subset \bigwedge \times M \times X$ such that $(\mu_\alpha, \lambda_\alpha, x_\alpha) \to (\mu_0, \lambda_0, x_0)$ and

$$g(\mu_\alpha, \lambda_\alpha, x_\alpha) \leq l, \quad \forall \alpha.$$

It follows that

$$\min_{t_\alpha \in T(y_\alpha, \lambda_\alpha), y_\alpha \in K(x_\alpha, \mu_\alpha)} \xi_e(x_\alpha, \langle t_\alpha, y_\alpha - x_\alpha \rangle) \leq l, \quad \forall x_\alpha \in E(\mu_\alpha). \tag{16}$$

Since K is B-u.s.c with compact values at (x_0, μ_0), $x_0 \in K(x_0, \mu_0)$. By (16), there exist $\bar{y}_\alpha \in K(x_\alpha, \mu_\alpha)$ and $\bar{t}_\alpha \in K(\bar{y}_\alpha, \lambda_\alpha)$ such that

$$\xi_e(x_\alpha, \langle \bar{t}_\alpha, \bar{y}_\alpha - x_\alpha \rangle) \leq l.$$

By the upper semicontinuity of K at (x_0, μ_0) and the compactness of $K(x_0, \mu_0)$, there exists $y_0 \in K(x_0, \mu_0)$ such that $\bar{y}_\alpha \to y_0$ (taking a subnet $\{\bar{y}_\beta\}$ of $\{\bar{y}_\alpha\}$ if necessary). Similarly, there exists $t_0 \in T(y_0, \lambda_0)$ such that $\bar{t}_\beta \to t_0$. From Proposition 2.4, it follows that ξ_e is lower semicontinuous. Then, we can conclude

$$\xi_e(x_0, \langle t_0, y_0 - x_0 \rangle) \leq \liminf_\alpha \xi_e(x_\alpha, \langle \bar{t}_\alpha, \bar{y}_\alpha - x_\alpha \rangle) \leq l.$$

Therefore, the level set $\{(\mu, \lambda, x) : g(\mu, \lambda, x) \leq l\}$ is closed-valued for all $l \in R$ and so, g is lower semicontinuous on $\bigwedge \times M \times X$. □

Lemma 4.8 *Let $E(\mu)$ be nonempty for each $\mu \in \bigwedge$. Assume that the following conditions are satisfied:*

(i) *$K(\cdot, \cdot)$ is B-continuous with compact values on $X \times \bigwedge$;*
(ii) *$T(\cdot, \cdot)$ is B-continuous with compact values on $X \times M$;*
(iii) *$C(\cdot)$ and $V(\cdot) = Y \setminus intC(\cdot)$ are B-u.s.c on X, and $e(\cdot) \in intC(\cdot)$ is continuous on X.*

Then, $g(\cdot, \cdot, \cdot)$ is continuous.

Proof By Lemma 4.7, we only need to prove that g is upper semicontinuous, i.e., $-g$ is lower semicontinuous.

It follows from (i) and (ii) that $g(\mu, \lambda, x)$ is finite for all $\mu \in \bigwedge, \lambda \in M$ and $x \in E(\mu)$. Taking $\iota \in R$. Assume that net $\{(\mu_\alpha, \lambda_\alpha, x_\alpha)\} \subset \bigwedge \times M \times X$ such that $(\mu_\alpha, \lambda_\alpha, x_\alpha) \to (\mu_0, \lambda_0, x_0)$ and

$$-g(\mu_\alpha, \lambda_\alpha, x_\alpha) \leq \iota, \quad \forall \alpha.$$

Therefore, one has

$$-\min_{t_\alpha \in T(y_\alpha, \lambda_\alpha), y_\alpha \in K(x_\alpha, \mu_\alpha)} \xi_e(x_\alpha, \langle t_\alpha, y_\alpha - x_\alpha \rangle) \leq \iota, \quad \forall x_\alpha \in E(\mu_\alpha),$$

that is,

$$\max_{t_\alpha \in T(y_\alpha, \lambda_\alpha), y_\alpha \in K(x_\alpha, \mu_\alpha)} -\xi_e(x_\alpha, \langle t_\alpha, y_\alpha - x_\alpha \rangle) \leq \iota, \quad \forall x_\alpha \in E(\mu_\alpha). \tag{17}$$

Since K is B-u.s.c with compact values at (x_0, μ_0), $x_0 \in K(x_0, \mu_0)$. From (17), it implies that

$$-\xi_e(x_\alpha, \langle t_\alpha, y_\alpha - x_\alpha \rangle) \leq \iota, \quad \forall y_\alpha \in K(x_\alpha, \mu_\alpha),\ t_\alpha \in T(y_\alpha, \lambda_\alpha).$$

By the Berge lower semicontinuity of K at (x_0, μ_0) and the compactness of $K(x_0, \mu_0)$, for any $y_0 \in K(x_0, \mu_0)$, there exists $y_\alpha \in K(x_\alpha, \mu_\alpha)$ such that $y_\alpha \to y_0$ (taking a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ if necessary). Since T is B-l.s.c with compact values at (y_0, λ_0), then for any $t_0 \in T(y_0, \lambda_0)$, there exists $t_\alpha \in T(y_\alpha, \lambda_\alpha)$ such that $t_\beta \to t_0$. From Proposition 2.4 and (iii), it follows that ξ_e is upper semicontinuous. Thus, we have

$$\begin{aligned}-\xi_e(x_0, \langle t_0, y_0 - x_0\rangle) &\le \liminf_{\alpha} -\xi_e(x_\alpha, \langle \bar{t}_\alpha, \bar{y}_\alpha - x_\alpha\rangle)\\ &\le \limsup_{\alpha} -\xi_e(x_\alpha, \langle \bar{t}_\alpha, \bar{y}_\alpha - x_\alpha\rangle) \le \iota.\end{aligned}$$

Then, the level set $\{(\mu, \lambda, x) : -g(\mu, \lambda, x) \le \iota\}$ is closed-valued for all $\iota \in R$, and so, $-g$ is lower semicontinuous. Therefore, g is upper semicontinuous and so continuous on $\bigwedge \times M \times X$. □

Motivated by the hypothesis (Zhao 1997; Kien 2005, (H_1); Li and Chen 2009; Chen et al. 2010, (Hg); Zhong and Huang 2011b, (Hg)$'$), using the parametric gap function g, we also introduce the following assumption:

(Hg)$''$ For any given $(\mu_0, \lambda_0) \in \bigwedge \times M$. For any $\epsilon > 0$, there exist $\rho > 0$ and $\delta > 0$ such that, for any $(\mu, \lambda) \in B((\mu_0, \lambda_0), \delta)$ and $x \in \Delta(\mu, \lambda, \epsilon) = E(\mu) \setminus U(S(\mu, \lambda), \epsilon)$, one has $g(\mu, \lambda, x) \le -\rho$.

Remark 4.9 It is easy to see that, if $\bigwedge$ is the same as M, and $\bigwedge$ is a metric space, $C(x) \equiv C$ for all $x \in X$, and $\mu = \lambda$, then the hypothesis (Hg)$''$ is reduced to the hypothesis (Hg)$'$ (Zhong and Huang 2011b) in the sense of presentation form.

Remark 4.10 As pointed out in Zhao (1997), the above hypothesis (Hg)$''$ is characterized by a common theme used in mathematical analysis. Such a theme interprets a proposition associated with a set in terms of other propositions related with the complement set. Instead of looking for restrictions within the solution set, the hypothesis (Hg)$''$ puts restrictions on the behavior of the parametric gap function on the complement of solution set. As showed in Chen et al. (2010), the hypothesis (Hg)$''$ seems to be reasonable in establishing the Hausdorff lower semicontinuity and Hausdorff continuity of $S(\cdot, \cdot)$ (see Li and Chen 2009; Zhao 1997; Chen et al. 2010; Zhong and Huang 2011b). But the hypothesis (Hg)$''$ includes the solution information $S(\mu, \lambda)$ for all (μ, λ) in a neighborhood of (μ_0, λ_0), which may not be verified. Inspired by Zhong and Huang (2011b), we give an equivalence formulation of (Hg)$''$.

Lemma 4.11 *Assume that all conditions in Lemma 4.8 are satisfied. Let $\epsilon > 0$ and $\phi_\epsilon(\mu, \lambda) = \sup_{x \in \Delta(\mu, \lambda, \epsilon)} g(\mu, \lambda, x)$. Then $(Hg)''$ holds iff, for every $\epsilon > 0$,* $\limsup_{(\mu,\lambda) \to (\mu_0, \lambda_0)} \phi_\epsilon(\mu, \lambda) < 0$.

Proof The proof is similar to that of Zhong and Huang (2011b, Lemma 2.8), and so it is omitted here. □

Theorem 4.12 *Assume that the following conditions are satisfied:*

(i) *$E(\cdot)$ is B-l.s.c with compact values on $\bigwedge$;*
(ii) *$K(\cdot, \cdot)$ is B-continuous with compact values on $X \times \bigwedge$;*
(iii) *$T(\cdot, \cdot)$ is B-continuous with compact values on $X \times M$;*
(iv) *$C(\cdot)$ is B-u.s.c on X, and $e(\cdot) \in intC(\cdot)$ is continuous on X;*
(v) *$W(\cdot) = Y \setminus -intC(\cdot)$ is closed on X.*

Then the solution mapping $S(\cdot, \cdot)$ is H-l.s.c on $\bigwedge \times M$ if and only if (Hg)$''$ holds.

Proof We first prove the sufficiency by the method of Chen et al. (2010). Assume that $(\mathrm{Hg})''$ holds. Suppose to the contrary that there exist some $(\mu_0, \lambda_0) \in \bigwedge \times M$ such that the solution mapping S is not H-l.s.c at (μ_0, λ_0). Then there exist a neighborhood B_0 of 0_X, and nets $\{(\mu_\alpha, \lambda_\alpha)\} \subset \bigwedge \times M$ with $(\mu_\alpha, \lambda_\alpha) \to (\mu_0, \lambda_0)$ and $\{x_\alpha\}$ such that

$$x_\alpha \in S(\mu_0, \lambda_0) \setminus (S(\mu_\alpha, \lambda_\alpha) + B_0). \tag{18}$$

By Corollary 3.4, $S(\mu_0, \lambda_0)$ is a compact set. Without loss of generality, assume that $x_\alpha \to x_0 \in S(\mu_0, \lambda_0)$. For B_0, there exists a balanced open neighborhood $\bigcirc(\epsilon)$ of 0_X, where $\epsilon > 0$, such that $\bigcirc(\epsilon) + \bigcirc(\epsilon) + \bigcirc(\epsilon) \subset B_0$. It is easy to check that

$$(x_0 + \bigcirc(\epsilon)) \cap E(\mu_0) \neq \emptyset, \quad \forall \epsilon > 0.$$

Since $E(\cdot)$ is B-l.s.c at μ_0, there exist some β_1 such that

$$(x_0 + \bigcirc(\epsilon)) \cap E(\mu_\beta) \neq \emptyset, \quad \forall \beta \geq \beta_1.$$

For a given $\epsilon \in (0, 1]$. Assume that $y_\beta \in (x_0 + \bigcirc(\epsilon)) \cap E(\mu_\beta)$. We assert that $y_\beta \notin S(\mu_\beta, \lambda_\beta) + \bigcirc(\epsilon)$. Suppose to the contrary that $y_\beta \in S(\mu_\beta, \lambda_\beta) + \bigcirc(\epsilon)$. Then there exists $z_\beta \in S(\mu_\beta, \lambda_\beta)$ such that $y_\beta - z_\beta \in \bigcirc(\epsilon)$. Note that $x_\alpha \to x_0 \in S(\mu_0, \lambda_0)$. Without loss of generality, we may assume that $x_\beta - x_0 \in \bigcirc(\epsilon)$, whenever β is sufficiently large. Therefore, one has

$$\begin{aligned} x_\beta - z_\beta &= (x_\beta - x_0) + (x_0 - y_\beta) + (y_\beta - z_\beta) \\ &\in \bigcirc(\epsilon) + \bigcirc(\epsilon) + \bigcirc(\epsilon) \subset B_0. \end{aligned}$$

This yields that $x_\beta \in S(\mu_\beta, \lambda_\beta) + B_0$, which contradicts (18). Thus, $y_\beta \notin S(\mu_\beta, \lambda_\beta) + \bigcirc(\epsilon)$. In the light of $(\mathrm{Hg})''$, there are two real numbers $\rho > 0$ and $\delta > 0$ such that, for any $(\mu_\beta, \lambda_\beta) \in B((\mu_0, \lambda_0), \delta)$, and $y_\beta \notin S(\mu_\beta, \lambda_\beta) + \bigcirc(\epsilon)$, one has

$$g(\mu_\beta, \lambda_\beta, y_\beta) \leq -\rho. \tag{19}$$

By Lemma 4.7, g is lower semicontinuous. So, for any $\Theta > 0$,

$$g(\mu_\beta, \lambda_\beta, y_\beta) \geq g(\mu_0, \lambda_0, y_0) - \Theta. \tag{20}$$

Without loss of generality, assume that $\Theta < \rho$. Then, from (19) and (20),

$$g(\mu_0, \lambda_0, y_0) \leq \Theta - \rho < 0,$$

that is,

$$g(\mu_0, \lambda_0, y_0) = \min_{t \in T(y_0, \lambda_0), y \in K(x_0, \mu_0)} \xi_e(x_0, \langle t, y - x_0 \rangle) < 0.$$

Hence, there exist $y_0 \in K(x_0, \mu_0) and t_0 \in T(y_0, \lambda_0)$ such that

$$\xi_e(x_0, \langle t_0, y_0 - x_0 \rangle) < 0.$$

From Proposition 2.3, it follows that

$$\langle t_0, y_0 - x_0 \rangle \in -\mathrm{int} C(x_0),$$

which contradicts that $x_0 \in S(\mu_0, \lambda_0)$. Hence, S is H-l.s.c on $\bigwedge \times M$.

Conversely, suppose to the contrary that S is H-l.s.c on $\bigwedge \times M$, but $(\mathrm{Hg})''$ is not true. By Lemmas 4.5 and 4.11, there exists $\epsilon_0 > 0$ such that

$$\limsup_{(\mu, \lambda) \to (\mu_0, \lambda_0)} \phi_{\epsilon_0}(\mu, \lambda) = 0.$$

Then, there is a sequence $\{(\mu_n, \lambda_n)\}$ with $(\mu_n, \lambda_n) \to (\mu_0, \lambda_0)$ as $n \to \infty$ and

$$\lim_{n\to\infty} \phi_{\epsilon_0}(\mu_n, \lambda_n) = \lim_{n\to\infty} \sup_{x\in\Delta(\mu_n,\lambda_n,\epsilon_0)} g(\mu_n, \lambda_n, x) = 0.$$

Since $\Delta(\mu_n, \lambda_n, \epsilon_0)$ is a compact set,

$$\lim_{n\to\infty} \phi_{\epsilon_0}(\mu_n, \lambda_n) = \lim_{n\to\infty} \max_{x\in\Delta(\mu_n,\lambda_n,\epsilon_0)} g(\mu_n, \lambda_n, x) = 0. \tag{21}$$

Then, there is $x_n \in \Delta(\mu_n, \lambda_n, \epsilon_0) = E(\mu_n) \setminus U(S(\mu_n, \lambda_n), \epsilon_0)$, such that

$$\phi_{\epsilon_0}(\mu_n, \lambda_n) = g(\mu_n, \lambda_n, x_n).$$

By Lemma 4.8, g is continuous on $\bigwedge \times M \times X$. This together with (21) yields that

$$0 = \lim_{n\to\infty} g(\mu_n, \lambda_n, x_n) = g(\mu_0, \lambda_0, x_0).$$

By virtue of (i), we may assume that $x_n \to x_0$ with $x_0 \in E(\mu_0)$. Then $x_0 \in S(\mu_0, \lambda_0)$. Since S is H-l.s.c on $\bigwedge \times M$, for any $z \in S(\mu_0, \lambda_0)$, there is a sequence $\{z_n\}$ with $z_n \in S(\mu_n, \lambda_n)$ for all n such that $z_n \to z$ as $n \to \infty$. In view of $x_n \in \Delta(\mu_n, \lambda_n, \epsilon_0)$, one has

$$\|z_n - x_n\| \geq \epsilon_0.$$

Moreover, we get

$$\epsilon_0 \leq \lim_{n\to\infty} \|z_n - x_n\| = \|z - x_0\|,$$

that is,

$$\|z - x_0\| \geq \epsilon_0 > 0, \quad \forall z \in S(\mu_0, \lambda_0).$$

This together with $x_0 \in S(\mu_0, \lambda_0)$ shows that

$$0 = \|x_0 - x_0\| \geq \epsilon_0 > 0,$$

which is a contradiction. □

Example 4.13 Let $\bigwedge = (-\frac{1}{2}, \frac{1}{2})$, $M = (-1, 1)$, $X = R$, $Y = R^2$, and let $C(x) = R^2_+$ and $e(x) = (1, 1)^{\mathrm{T}} \in \mathrm{int}R^2_+$ for all $x \in X$. Define the set-valued mappings $K : X \times \bigwedge \to 2^X$ and $T : X \times M \to 2^Y$ by, for any $x \in X$, $\mu \in M$ and $\lambda \in \bigwedge$,

$$T(x, \lambda) =: \{(1, \ell)^{\mathrm{T}} : \ell \in [1, 1 + \lambda^2]\}, \quad K(x, \mu) =: \left[\frac{x - \mu}{2}, 1 - \mu\right].$$

Then, we have

$$E(\mu) = \{x \in X : x \in K(x, \mu)\} = [-\mu, 1 - \mu], \quad \forall \mu \in \bigwedge.$$

Clearly, the conditions (i)–(v) of Theorem 4.12 are satisfied. By computation, we get $S(\mu, \lambda) = \{-\mu\}$ for all $(\mu, \lambda) \in \bigwedge \times M$. So, $S(\cdot, \cdot)$ is H-l.s.c on $\bigwedge \times M$.

Next, we verify the assumption (Hg)$''$. Since $e(x) = (1,1)^{\mathrm{T}} \in \mathrm{int}R^2_+$ for all $x \in X$, then, from Example 2.2, for any $(\mu, \lambda) \in \bigwedge \times M$ and $x \in E(\mu)$,

$$\begin{aligned} g(\mu,\lambda,x) &= \min_{y\in K(x,\mu),t\in T(y,\lambda)} \xi_e(x,\langle t, y-x\rangle) \\ &= \min_{y\in K(x,\mu),t=(1,\ell)^{\mathrm{T}}\in T(y,\lambda)} \max\{y-x,\ell(y-x)\} \\ &= \min_{y\in[\frac{x-\mu}{2},1-\mu],1\le\ell\le 1+\lambda^2} \max\{y-x,\ell(y-x)\} \\ &= -\frac{x+\mu}{2}. \end{aligned}$$

Clearly, $g(\mu_0, \lambda_0, x_0) = 0$ if and only if $x_0 \in S(\mu_0, \lambda_0)$, and so

$$g(\mu,\lambda,x) = -\frac{x+\mu}{2} \le -\frac{-\mu+\mu}{2} = 0, \quad \forall x \in E(\mu).$$

Therefore, $g(\cdot,\cdot,\cdot)$ is a parametric gap function for (PGMVQVI).

For any given $(\bar{\mu}, \bar{\lambda}) \in \bigwedge \times M$ and $0 < \epsilon < 1$. Put $\rho = \frac{\epsilon}{3}$ and $0 < \delta < \rho$, we have, for any $(\mu, \lambda) \in B((\bar{\mu}, \bar{\lambda}), \delta)$ and $x \in \Delta(\mu, \lambda, \epsilon) = E(\mu) \setminus U(S(\mu, \lambda), \epsilon) = [\epsilon - \mu, 1 - \mu]$,

$$g(\mu,\lambda,x) \le -\frac{\epsilon-\mu+\mu}{2} = -\frac{\epsilon}{2} \le -\rho.$$

Hence, the assumption (Hg)$''$ is valid.

Example 4.14 Let $\bigwedge = M = (-1, 1)$, $X = R$, $Y = R^2$, and let $C(x) = R^2_+$ for all $x \in X$. Define the set-valued mappings $K : X \times \bigwedge \to 2^X$ and $T : X \times M \to 2^Y$ by

$$T(x,\lambda) =: \{(2,|\lambda|)^{\mathrm{T}}\}, K(x,\mu) =: [-1,1], \forall x \in X, \mu \in M, \lambda \in \bigwedge.$$

It is easy to see that the conditions (i)–(v) of Theorem 4.12 are satisfied. From simple computation, one has $E(\mu) = [-1, 1]$, for all $\mu \in M$, and so

$$S(\mu,\lambda) = \begin{cases} [-1,1], & \text{if } \lambda = 0, \\ \{-1\}, & \text{if } \lambda \neq 0. \end{cases}$$

Therefore, $S(\cdot,\cdot)$ is not H-l.s.c at $(\mu, 0)$, where $\mu \in (-1, 1)$. Let us show that the assumption (Hg)$''$ fails to hold at $(0, 0)$. Set $e(x) = (1,1)^{\mathrm{T}} \in \mathrm{int}R^2_+$. Then

$$\begin{aligned} g(\mu,\lambda,x) &= \min_{y\in K(x,\mu),t\in T(y,\lambda)} \xi_e(x,\langle t, y-x\rangle) \\ &= \min_{y\in[-1,1]} \max\{2(y-x),|\lambda|(y-x)\}. \end{aligned}$$

It is easy to see that g is a parametric gap function for (PGMVQVI). Since

$$g(\mu,\lambda,x) \le \max\{2(x-x),|\lambda|(x-x)\} = 0, \quad \forall x \in E(\mu),$$

and $g(\mu, \lambda, x) = 0$ for all $x \in S(\mu, \lambda)$. Taking some ϵ with $0 < \epsilon < 1$. For any $\rho > 0$, set $(\mu_n, \lambda_n) \to (0, 0)$ with $0 < |\lambda_n| < \frac{1}{2}\rho$ and $x_n = 1 \in \Delta(\mu_n, \lambda_n, \epsilon) = E(\mu_n) \setminus U(S(\mu_n, \lambda_n), \epsilon)$, $n \in N$,

$$g(\mu_n,\lambda_n,x_n) = |\lambda_n|(-1-x_n) = -2|\lambda_n| > -\rho.$$

Hence, the assumption (Hg)$''$ fails to hold at $(0, 0)$.

If $C(x) = C$ for all $x \in X$ and $e \in \text{int}C$, then from Theorem 4.12, we obtain the following corollary.

Corollary 4.15 *Assume that the following conditions are satisfied:*

(i) $E(\cdot)$ *is B-l.s.c with compact values on* $\bigwedge$;
(ii) $K(\cdot, \cdot)$ *is B-continuous with compact values on* $X \times \bigwedge$;
(iii) $T(\cdot, \cdot)$ *is B-continuous with compact values on* $X \times M$.*Then the solution mapping* $S(\cdot, \cdot)$ *is H-l.s.c on* $\bigwedge \times M$ *if and only if (Hg)*$''$ *holds.*

From Lemma 2.8 and Theorems 3.1 and 4.12, we can get the following result.

Corollary 4.16 *Assume that the following conditions are satisfied:*

(i) $E(\cdot)$ *is B-continuous with compact values on* $\bigwedge$;
(ii) $K(\cdot, \cdot)$ *is B-continuous with compact values on* $X \times \bigwedge$;
(iii) $T(\cdot, \cdot)$ *is B-continuous with compact values on* $X \times M$;
(iv) $C(\cdot)$ *is B-u.s.c on* X, *and* $e(\cdot) \in intC(\cdot)$ *is continuous on* X;
(v) $W(\cdot) = Y \setminus -intC(\cdot)$ *is closed on* X.

Then the solution mapping $S(\cdot, \cdot)$ *is B-continuous and H-continuous if and only if (Hg)*$''$ *holds.*

Remark 4.17 The condition (ii) of Theorem 4.12 and Corollary 4.15 may not imply the Berge lower semicontinuity of $E(\cdot)$.

Example 4.18 Let $\bigwedge = [-1, 1]$ and $X = R$. Define a set-valued mapping $K : X \times \bigwedge \to 2^X$ by

$$K(x, \mu) = \begin{cases} \{|\mu|\chi : \chi \in X, x \leq \chi \leq -x\}, & \text{if } x \leq -1, \\ \{|\mu|\chi : \chi \in X, -1 \leq \chi \leq 1\}, & \text{if } -1 < x < 1, \\ \{|\mu|\chi : \chi \in X, -x \leq \chi \leq x\}, & \text{if } x \geq 1. \end{cases}$$

Obviously, $K(\cdot, \cdot)$ is B-continuous with compact values on $X \times \bigwedge$. But the set-valued mapping

$$\begin{aligned} E(\mu) &= \{x \in X : x \in K(x, \mu)\} \\ &= \begin{cases} R, & \text{if } \mu = 1, -1, \\ \{\chi : \chi \in X, -|\mu| \leq \chi \leq |\mu|\}, & \text{if } -1 < \mu < 1, \end{cases} \end{aligned}$$

is not B-l.s.c at $\mu = 1$, or $\mu = -1$.

5 An application

In this section, let the Hausdorff topological space $\bigwedge$ be the same as M, A be a nonempty, closed and convex subset of a locally convex Hausdorff topological vector spaces X, and $C(x) = C$ be closed convex cone with nonempty interior for all $x \in X$. We discuss the solution stability for a class of parametric vector quasivariational inequality of the Minty type.

Consider the following parametric vector quasivariational inequality problem [in short (PVQI)]: find $x \in K(x, \mu) \cap A$ such that

$$\langle t, y - x \rangle \notin -\text{int}C, \quad \forall y \in K(x, \mu),\ t \in T(y, \mu). \tag{22}$$

Denote the solution set of (PVQI) by $S(\mu)$. Assume that $K(x,\mu) \cap A \neq \emptyset$ for all $(x,\mu) \in X \times \bigwedge$.

If $C = R_+$, $\bigwedge$ is a nonempty closed subset of R^n and A is a nonempty, closed and convex subset of R^m, $K : R^m \times \bigwedge \to 2^{R^m}$ and $T : R^m \times \bigwedge \to 2^{R^m}$ are both set-valued mappings with K being a closed-valued mapping, then (PVQI) reduces to the following parametric quasivariational inequality of the Minty type [in short (PQI)] corresponding to a parameter $\mu_0 \in \bigwedge$: find $x_0 \in K(x_0,\mu_0) \cap A$ such that

$$\langle t, x_0 - y\rangle \leq 0, \quad \forall y \in K(x_0,\mu_0),\ t \in T(y,\mu_0), \tag{23}$$

which was considered by Lalitha and Bhatia (2011).

Theorem 5.1 *Let A be a nonempty compact subset of X. For $\mu_0 \in \bigwedge$, if the following conditions are satisfied:*

(i) *$E(\cdot)$ is B-u.s.c with compact values at μ_0;*
(ii) *For any $x_0 \in K(x_0,\mu_0)$, $(x_\alpha,\mu_\alpha) \to (x_0,\mu_0)$ and*

$$\langle t_0, y_0 - x_0\rangle \in -intC \quad \textit{for some} \quad y_0 \in K(x_0,\mu_0), t_0 \in T(y_0,\mu_0)$$

implies that there exists α such that

$$\langle t_\alpha, y_\alpha - x_\alpha\rangle \in -intC \quad \textit{for some} \quad y_\alpha \in K(x_\alpha,\mu_\alpha), t_\alpha \in T(y_\alpha,\mu_\alpha).$$

Then the solution mapping $S(\cdot)$ is B-u.s.c at μ_0. Further, $S(\cdot)$ is H-u.s.c at μ_0.

Proof It directly follows from Theorem 8 of Aubin and Ekeland (1984, Sect. 1, Chap. 3) and Theorem 3.5. □

Theorem 5.2 *Let A be a nonempty compact subset of X. For $\mu_0 \in \bigwedge$, if the following conditions are satisfied:*

(i) *$E(\cdot)$ is B-u.s.c with compact values at μ_0;*
(ii) *$K(\cdot,\cdot)$ is B-l.s.c on $X \times \{\mu_0\}$;*
(iii) *$T(\cdot,\cdot)$ is B-l.s.c with compact values on $X \times \{\mu_0\}$.*

Then the solution mapping $S(\cdot)$ is B-u.s.c at μ_0. Moreover, $S(\cdot)$ is H-u.s.c at μ_0.

Proof It directly follows from Theorem 8 of Aubin and Ekeland (1984, Sect. 1, Chap. 3) and Theorem 3.1. □

Remark 5.3 Theorem 5.1 extends Theorem 3.2 of Lalitha and Bhatia (2011) in the following aspects:

(i) The underlying space and the objective space are both extended from R^n and (R^m) to Hausdorff topological vector spaces, respectively.
(ii) The parametric quasivariational inequality problem (PQI) is extended to the parametric vector quasivariational inequality problem (PVQI).

6 Conclusions

In this paper, under some suitable assumptions, the Berge (Hausdorff) upper semicontinuity, Berge lower semicontinuity and closedness of the solution set mapping for the (PGMVQVI) are established in Hausdorff topological vector spaces. Secondly, a parametric gap function is introduced for the (PGMVQVI). Using the assumption (Hg)$''$ and parametric gap function, sufficient and necessary conditions of the Hausdorff lower semicontinuity and Hausdorff continuity of the solution set mapping for the (PGMVQVI) are derived without monotonicity. In the end, we also obtain the Berge (Hausdorff) upper semicontinuity of the solution set mapping for a (PVQI), which is a generalization of the model discussed in Lalitha and Bhatia (2011). Referees point out that our results can be applied to general variational inequalities (Noor 1998, 2000, 2004, 2006). In further research, we may study the following two questions:

Question I If the solution set mapping S of the (PGMVQVI) is B-u.s.c (H-u.s.c) at some point $(\mu, \lambda) \in \bigwedge \times M$, does the condition (ii) of Theorem 3.5 hold?

Question II If the solution set mapping S of the (PGMVQVI) is B-l.s.c at some point $(\mu, \lambda) \in \bigwedge \times M$, does the condition (ii) of Theorem 4.1 hold?

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