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Bilevel invex equilibrium problems with applications

Jia-wei Chen · Zhongping Wan · Yun-Zhi Zou

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Abstract In this paper, bilevel invex equilibrium problems of Hartman-Stampacchia type and Minty type [resp., in short, (HSBEP) and (MBEP)] are firstly introduced in finite Euclidean spaces. The relationships between (HSBEP) and (MBEP) are presented under some suitable conditions. By using fixed point technique, the nonemptiness and compactness of solution sets to (HSBEP) and (MBEP) are established under the invexity, respectively. As applications, we investigate the existence of solution and the behavior of solution set to the bilevel pseudomonotone variational inequalities of [Anh et al. J Glob Optim 2012, doi:10.1007/s10898-012-9870-y] and the solvability of minimization problem with variational inequality constraint.

Keywords Bilevel invex equilibrium problem · Bilevel variational inequalities · Minimization problem with variational inequality constraint · Existence · KKM mapping

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J. Chen (✉) · Z. Wan
School of Mathematics and Statistics, Wuhan University,
Wuhan 430072, Hubei, People's Republic of China
e-mail: J.W.Chen713@163.com

J. Chen
School of Information and Mathematics, Yangtze University,
Jingzhou 434023, Hubei, People's Republic of China

Y.-Z. Zou
Department of Mathematics, Sichuan University,
Chengdu 610064, Sichuan, People's Republic of China

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1 Introduction

The equilibrium problem, which was first introduced by Blum and Oettli [5], provides a unified model of many problems such as optimization problems, variational inequality problems, complementarity problems, fixed point problems and so on. Subsequently, equilibrium and generalized different types of equilibrium problems were intensively studied (see, e.g., [3,4,7–10,20] and the references therein). Let $\mathcal{H}$ be a Hilbert space and D be a nonempty closed subset of $\mathcal{H}$, $P : D \times D \to R$ be a continuous bifunction. Noor [27,28] introduced and studied the following invex equilibrium (equilibrium-like) problem: find $u \in D$ such that

$$P(u, \eta(v, u)) \geq 0, \quad \forall v \in D,$$

where the mapping $\eta : D \times D \to D$. Thereafter, Noor et al. [29,30] further studied the invex equilibrium (equilibrium-like) problems in different ways. They pointed out that the invex equilibrium problem contained variational-like inequalities, equilibrium problems and variational inequalities as special cases. Guu and Li [19] also studied this class of problem in the setting of the vector-valued case, but they did not use the term *invex equilibrium problem*. The relationships between vector variational-like inequalities and vector optimization problems were established under the some suitable assumptions. That is, the invex equilibrium problem is closely related to the optimization problem.

For the past decades, mathematical programs with variational inequality, equilibrium and complementarity constraints have attracted many scholars' interests (see, e.g., [4,18,22–25,31,32] and references therein). In 2010, Moudafi [26] introduced a class of bilevel equilibrium problem [for short, (BEP)]: find $x \in S_F$ such that

$$H(x, y) \geq 0, \quad \forall y \in S_F,$$

where S_F is the solution set of the following equilibrium problem: find $u \in K$ such that

$$F(u, y) \geq 0, \quad \forall y \in K,$$

where K is a nonempty closed convex subset of a Hilbert space, and $H, F : K \times K \to R$ are two functions. He pointed out that this class is interesting since it includes hierarchical optimization problems, optimization problems with equilibrium, variational inequalities and complementarity constraints as special cases. By using the proximal method, an iterative algorithm to compute approximate solution of the (BEP) and the weak convergence of the iterative sequence generated by the algorithm were suggested and derived, respectively. Motivated by Moudafi's works [26], Ding [11,12] considered the following *bilevel mixed equilibrium problem* [for short, (BMEP)] [(1.1),(1.2)] in reflexive Banach spaces: find $x \in S_{\Psi,\psi}$ such that

$$h(x, y) + \phi(y, x) - \phi(x, x) \geq 0, \quad \forall y \in S_{f,\psi}, \tag{1.1}$$

where $S_{f,\psi}$ is the solution set of the following mixed equilibrium problem: find $y \in K$ such that

$$f(y,z) + \psi(z,y) - \psi(y,y) \geq 0, \quad \forall z \in K, \tag{1.2}$$

where E is a real Banach space with its dual space E^*, the norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed convex subset of E, $h, f : K \times K \to R \cup \{+\infty\}$ and let $\phi, \psi : E \times E \to R \cup \{+\infty\}$ be functions. Ding [11,12] studied the existence of solution and iterative algorithms for (BMEP) [(1.1), (1.2)]. Since then, Anh et al. [1,2], Chadli, Mahdioui and Yao[6], Ding [13,14], Liou and Yao [15] and Dinh and Muu [16] continued to study the (BEP) on existence, well-posedness and algorithm aspects in the setting of convexity. However, in many cases, it is difficult to ensure the convexity.

Throughout this paper, let R^n be n-dimensional Euclidean space, K be a nonempty subset of R^n, $\mathcal{K}$ a closed convex subset of R^n with $K \subseteq \mathcal{K}$, and let $\eta : \mathcal{K} \times \mathcal{K} \to R^n$, $\Phi, \Psi : K \times K \to R \cup \{+\infty\}$ be functions. We denote co and clco by the convex hull and the closed convex hull, respectively.

Motivated and inspired by the above works, we consider the following *bilevel invex equilibrium problem of Hartman-Stampacchia type* [for short, (HSBEP)]: find $x \in S_{HS}$ such that

$$\Phi(x, \eta(y,x)) \geq 0, \quad \forall y \in S_{HS}, \tag{1.3}$$

where S_{HS} is the solution set of the lower level invex equilibrium problem: find $y^* \in K$ such that

$$\Psi(y^*, \eta(z, y^*)) \geq 0, \quad \forall z \in K, \tag{1.4}$$

where K is an invex subset of R^n.

Denote the solution set of (HSBEP) [(1.3), (1.4)] by $\aleph_{HS}$.

We also consider the following *bilevel invex equilibrium problem of Minty type* [for short, (MBEP)]: find $x \in S_M$ such that

$$\Phi(y, -\eta(y,x)) \leq 0, \quad \forall y \in S_M, \tag{1.5}$$

where S_M is the solution set of the lower level mixed equilibrium problem: find $y^* \in K$ such that

$$\Psi(z, -\eta(z, y^*)) \leq 0, \quad \forall z \in K. \tag{1.6}$$

Denote the solution set of (MBEP) [(1.5), (1.6)] by $\aleph_M$.

We first recall some definitions and lemmas which are needed in the main results of this work.

Definition 1.1 [28] K is said to be η-connected if, for any $x, y \in K$ and $t \in [0,1]$, $x + t\eta(y,x) \in K$.

Remark 1.1 If K is η-connected, we also say K is an invex set with respect to η. Moreover, any convex set is an invex set with respect to $\eta(x,y) = x - y$.

Definition 1.2 The bifunction Ψ is said to be

(i) η-pseudomonotone on K if, for any $x, y \in K$,

$$\Psi(y, \eta(x, y)) \geq 0 \Rightarrow \Psi(x, -\eta(x, y)) \leq 0;$$

(ii) strictly η-pseudomonotone on K if, for any $x, y \in K$,

$$\Psi(y, \eta(x, y)) \geq 0 \Rightarrow \begin{cases} \Psi(x, -\eta(x, y)) < 0, & \text{if } \ x \neq y, \\ \Psi(x, -\eta(x, y)) \leq 0, & \text{otherwise.} \end{cases}$$

It is easy to see that the strict η-pseudomonotonicity implies the η-pseudomonotonicity. But the converse is not true. If $\eta(x, y) = x - y$ for all $x, y \in R^n$, then the η-pseudomonotonicity is reduced to the pseudomonotonicity of Lalitha and Mehta [21].

Example 1.1 Let $K = [0, 1]$, and let $\eta(x, y) = \cos x - \cos y$, $\Psi(y, \eta(x, y)) = \langle \sin y - 1, \cos x - \cos y\rangle$ for all $x, y \in K$. It is easy to verify that $\Psi(y, \eta(x, y)) = \langle \sin y - 1, \cos x - \cos y\rangle \geq 0$ whenever $y \leq x$. Note that for each $x, y \in K$, $y \leq x$, $\sin x - 1 < 0$ and $\cos y - \cos x \geq 0$. Then

$$\Psi(x, -\eta(x, y)) = \langle \sin x - 1, \cos y - \cos x\rangle \leq 0, \quad \forall x, y \in K, y \leq x.$$

This implies that Ψ is η-pseudomonotone on K. Clearly, Ψ is also strictly η-pseudomonotone on K. Indeed, if $x, y \in K$, $y < x$, then $\sin x - 1 < 0$, $\cos y - \cos x > 0$ and so,

$$\Psi(x, -\eta(x, y)) = \langle \sin x - 1, \cos y - \cos x\rangle < 0.$$

Example 1.2 Let $K = [-1, 1]$ and let η, Ψ be the same as Example 1.1. It is easy to verify that $\Psi(y, \eta(x, y)) = \langle \sin y - 1, \cos x - \cos y\rangle \geq 0$ whenever $|y| \leq |x|$. Note that for each $x, y \in K$, $|y| \leq |x|$, $\sin x - 1 < 0$ and $\cos y - \cos x \geq 0$. Then

$$\Psi(x, -\eta(x, y)) = \langle \sin x - 1, \cos y - \cos x\rangle \leq 0, \quad \forall x, y \in K, |y| \leq |x|.$$

Therefore, Ψ is η-pseudomonotone on K. But Ψ is not strictly η-pseudomonotone on K. Indeed, if $y = -1 < x = 1$, then $\sin x - 1 < 0$, $\cos y - \cos x = 0$ and so,

$$\Psi(x, -\eta(x, y)) = \langle \sin x - 1, \cos y - \cos x\rangle = 0.$$

Definition 1.3 [19] The function $x \mapsto \eta(x, \cdot)$ is said to be

(i) R^n_+-convex if, for any $x_j \in \mathcal{K}$ $(j = 1, 2, \ldots, m)$ and $t_j \in [0, 1]$ with $\sum_{j=1}^m t_j = 1$,

$$\eta\left(\sum_{j=1}^m t_j x_j, \cdot\right) \in \sum_{j=1}^m t_j \eta(x_j, \cdot) - R^n_+;$$

(ii) affine if, for any $x_j \in \mathcal{K}$ $(j = 1, 2, \ldots, m)$ and $t_j \in [0, 1]$ with $\sum_{j=1}^m t_j = 1$,

$$\eta\left(\sum_{j=1}^m t_j x_j, \cdot\right) = \sum_{j=1}^m t_j \eta(x_j, \cdot).$$

Definition 1.4 [19] Let K be η-connected. The bifunction Ψ is said to be η-hemicontinuous on K if, for any $x, y \in K$ and $t \in [0, 1]$, the mapping $t \mapsto \Psi(x + t\eta(y, x), -\eta(y, x))$ is continuous at 0^+.

Remark 1.2 If $\eta(x, y) = x - y$ for all $x, y \in \mathcal{K}$, then the η-hemicontinuity reduces to hemicontinuity.

Definition 1.5 [19] The bifunction Ψ is said to be

(i) subodd if, for any $x \in K$ and $d \in R^n$,

$$\Psi(x, d) + \Psi(x, -d) \geq 0;$$

(ii) generalized subodd if, for any $x \in K$ and $d_j \in R^n (j = 1, 2, \ldots, m)$ with $\sum_{j=1}^m d_j = 0$,

$$\sum_{j=1}^m \Psi(x, d_j) \geq 0.$$

It easily follows from Definition 1.5 that if Ψ is generalized subodd, then Ψ is subodd. For example, we put $\Psi(x, y) = \langle e^x, y\rangle$ for all $x, y \in R^n$. Then Ψ is generalized subodd.

Definition 1.6 [30] Let K be η-connected (i.e., invex with respect to η). A function $f : K \to R$ is said to be invex with respect to η if,

$$f(y_1 + t\eta(y_2, y_1)) \leq tf(y_2) + (1 - t)f(y_1) \quad \forall y_1, y_2 \in K, t \in (0, 1).$$

Definition 1.7 [24] Let $g : R^n \to R$. g is said to be positively homogeneous if, $g(\lambda x) = \lambda g(x)$ for all $x \in R^n$ and $\lambda > 0$.

Definition 1.8 [17] A set-valued mapping $T : K \to 2^{R^n}$ is said to be

(i) closed if its graph, denoted by $Gr(T) = \{(x, \zeta) \in K \times R^n : \zeta \in T(x)\}$, is closed in $R^n \times R^n$;
(ii) KKM mapping if, for each finite subset $\{x_1, x_2, \ldots, x_m\}$ of K, $\mathrm{co}\{x_1, x_2, \ldots, x_m\}$ is contained in $\bigcup_{j=1}^{m} T(x_j)$, where co denotes the convex hull.

Lemma 1.1 (Fan-KKM theorem [17]) *Let $T : K \to 2^{R^n}$ be a KKM mapping such that for any $x \in K$, $T(x)$ is closed and $T(x^*)$ is bounded for some $x^* \in K$. Then there exists $y^* \in K$ such that $y^* \in T(x)$ for all $x \in K$, i.e., $\bigcap_{x \in K} T(x) \neq \emptyset$.*

2 Main results

In this section, we shall investigate the relationships between (HSBEP) and (MBEP), and the existence of solutions to (HSBEP) and (MBEP) under some suitable conditions.

Lemma 2.1 *(i) If Ψ is η-pseudomonotone on K, then $S_{HS} \subseteq S_M$;*
(ii) If K is an invex set with respect to η, $\eta(y + t\eta(z, y), y) = t\eta(z, y)$ for any $y, z \in K$ and $t \in (0, 1)$ and Ψ is positively homogeneous with respect to the second argument, subodd and η-hemicontinuous on K, then $S_M \subseteq S_{HS}$.

Proof (i) By the η-pseudomonotonicity of Ψ, we can derive the desired result.
(ii) Assume that $y^* \in S_M$. Then $y^* \in K$ and

$$\Psi(z, -\eta(z, y^*)) \leq 0, \quad \forall z \in K.$$

Since K is an invex set with respect to η, we obtain that $z + t\eta(z, y^*) \in K$ for all $z \in K$ and $t \in (0, 1)$. This, together with $\eta(y^* + t\eta(z, y^*), y^*) = t\eta(z, y^*)$ and positive homogeneity of Ψ with respect to the second argument, yields that

$$\begin{aligned} 0 &\geq \Psi(y^* + t\eta(z, y^*), -\frac{1}{t}\eta(y^* + t\eta(z, y^*), y^*)) \\ &= \Psi(y^* + t\eta(z, y^*), -\eta(z, y^*)). \end{aligned} \tag{2.1}$$

By the η-hemicontinuity of Ψ, taking the limit in (2.1) as $t \to 0^+$, one has

$$\Psi(y^*, -\eta(z, y^*)) \leq 0. \tag{2.2}$$

In view of the suboddness of Ψ, we get

$$\Psi(y^*, \eta(z, y^*)) + \Psi(y^*, -\eta(z, y^*)) \geq 0$$

and from (2.2),

$$\Psi(y^*, \eta(z, y^*)) \geq 0 \quad \forall z \in K.$$

Therefore, $y^* \in S_{HS}$. This completes the proof. □

Remark 2.1 Compared with Theorem 2.1 of Guu and Li [19], the condition $\eta(z, z) = 0$ for all $z \in K$ is removed.

The following result is a direct consequence of Lemma 2.1.

Corollary 2.1 *Let K be an invex set with respect to η and $\eta(y+t\eta(z, y), y) = t\eta(z, y)$ for all $y, z \in K$ and $t \in (0, 1)$, and let Ψ be subodd, η-pseudomonotone and η-hemicontinuous on K. Then $S_{HS} = S_M$.*

Lemma 2.2 *Assume that all conditions of Corollary 2.1 are satisfied. If Φ is subodd, η-pseudomonotone and η-hemicontinuous on K. Then (HSBEP)* [(1.3), (1.4)] *is equivalent to (MBEP)* [(1.5), (1.6)].

Proof From Corollary 2.1, one has $S_{HS} = S_M$. The rest of the proof is similar to the proof of Lemma 2.1 and so it is omitted here. This completes the proof.

Theorem 2.1 *Let $K \subseteq R^n$ be a nonempty closed and invex set with respect to η with $\eta(x, x) = 0$ for all $x \in \mathcal{K}$ and $\eta(y + t\eta(z, y), y) = t\eta(z, y)$ for all $y, z \in K$ and $t \in (0, 1)$. Assume that the following conditions hold:*

(i) *η is affine with respect to the first argument, and continuous with respect to the second argument;*
(ii) *Φ and Ψ are positively homogeneous and continuous with respect to the second argument, and η-pseudomonotone, generalized subodd and η-hemicontinuous on $\mathcal{K}$;*
(iii) *For each $y, z \in K$, the functions $\Phi(y, -\eta(y, \cdot))$ and $\Psi(z, -\eta(z, \cdot))$ are invex with respect to the η on K;*
(iv) *There exists a nonempty closed bounded convex set $\Omega \subseteq K$ such that for each $y \in \mathcal{K}\backslash\Omega$, there exists $z \in \Omega$ that satisfies $\Psi(z, -\eta(z, y)) > 0$.*
(v) *Further, assume that there exists a nonempty closed bounded convex set $\Xi \subseteq S_{HS}$ such that for each $x \in S_{HS}\backslash\Xi$, there exists $\tilde{y} \in \Xi$ that satisfies $\Phi(\tilde{y}, -\eta(\tilde{y}, x)) > 0$.*

Then $\aleph_{HS}$ and $\aleph_M$ are two nonempty compact and invex sets with respect to η.

Proof From Corollary 2.1 and Lemma 2.2, one has $S_{HS} = S_M$ and $\aleph_{HS} = \aleph_M$. This, together with (iv) and (v), implies that $S_{HS} = S_M \subseteq \Omega$ and $\aleph_{HS} = \aleph_M \subseteq \Xi$.

Let us show that S_{HS} and S_M are nonempty and closed. For any sequence $\{y_m\} \subseteq S_M$ with $y_m \to y_0$, we have $y_m \in K$ and

$$\Psi(z, -\eta(z, y_m)) \leq 0, \quad \forall z \in K. \tag{2.3}$$

By the continuity of η and Ψ with respect to the second argument, and from (2.3), one has $y_0 \in K$ and

$$\Psi(z, -\eta(z, y_0)) \leq 0, \quad \forall z \in K,$$

that is, $y_0 \in S_M$. Therefore, S_M and S_{HS} are closed and so are compact. In order to prove that $S_M \neq \emptyset$, we define a set-valued mapping $H : K \to 2^K$ by

$$H(z) = \{y \in \Omega : \Psi(z, -\eta(z, y)) \leq 0\}, \quad \forall z \in K.$$

Similarly, for each $z \in K$, $H(z)$ is closed and so is bounded, since Ω is bounded. Moreover, for each $z \in K$, $H(z)$ is compact. Clearly, $S_{HS} = S_M = \bigcap_{z \in K} H(z)$. We only need to prove that $\bigcap_{z \in K} H(z) \neq \emptyset$. It is sufficient to show that $\{H(z)\}_{z \in K}$ satisfies the finite intersection property. Let $\{z_1, z_2, \dots, z_m\} \subseteq K$ and $L = \mathrm{clco}[\{z_1, z_2, \dots, z_m\} \cup \Omega]$. Since Ω is a nonempty closed bounded convex subset of K, L is a nonempty closed bounded convex set, that is, L is a nonempty compact and convex set. Again from the closedness and convexity of $\mathcal{K}$ and $K \subseteq \mathcal{K}$, one has $L \subseteq \mathcal{K}$.

Define two set-valued mappings $E, F : L \to 2^L$ by, respectively,

$$E(z) = \{y \in L : \Psi(z, -\eta(z, y)) \leq 0\}, \quad \forall z \in L \tag{2.4}$$

and

$$F(z) = \{y \in L : \Psi(y, \eta(z, y)) \geq 0\}, \quad \forall z \in L. \tag{2.5}$$

By the η-pseudomonotonicity of Ψ, we can get that $E(z) \supseteq F(z)$ for all $z \in L$. In the light of $\eta(z, z) = 0$ for all $z \in K$, from the positive homogeneity of Ψ with respect to the second argument, one has

$$\Psi(z, -\eta(z, z)) = \Psi(z, 0) = \Psi(z, -\lambda\eta(z, z)) = \lambda\Psi(z, -\eta(z, z)), \quad \forall z \in L, \lambda > 0, \lambda \neq 1.$$

Thus, $\Psi(z, -\eta(z, z)) = \Psi(z, 0) = 0$. This implies that for each $z \in L, z \in E(z)$. Similarly, for each $z \in L, z \in F(z)$. Therefore, $E(z) \neq \emptyset$ and $F(z) \neq \emptyset$ for all $z \in L$. Next, we show that F is a KKM mapping. Suppose that there exist a finite subset $\{z_1, z_2, \dots, z_m\}$ of L and $z^* \in \mathrm{co}\{z_1, z_2, \dots, z_m\}$, which means that there exist $t_j \geq 0 (j = 1, 2, \dots, m)$ with $\sum_{j=1}^m t_j = 1$ that satisfy $z^* = \sum_{j=1}^m t_j z_j$, such that $z^* \notin \bigcup_{j=1}^m F(z_j)$. Since L is a convex subset of $\mathcal{K}$, $z^* \in L$. By (2.5), one has

$$\Psi(z^*, \eta(z_j, z^*)) < 0, \quad j = 1, 2, \dots, m.$$

Moreover, we have

$$\sum_{j=1}^m t_j \Psi(z^*, \eta(z_j, z^*)) < 0.$$

Again, from the positive homogeneity of Ψ with respect to the second argument and affinity of η with respect to the first argument, we obtain

$$\Psi(z^*, 0) = \Psi(z^*, \eta(z^*, z^*)) = \Psi\left(z^*, \sum_{j=1}^m t_j \eta(z_j, z^*)\right) = \sum_{j=1}^m t_j \Psi(z^*, \eta(z_j, z^*)) < 0,$$

which contradicts $\Psi(z^*, 0) = 0$. So, F is a KKM mapping. In view of $E(z) \supseteq F(z)$ for all $z \in L$, we conclude that E is also a KKM mapping. Since η and Ψ are continuous with respect to the second argument, for each $z \in L$, $E(z)$ is closed and so is bounded. Then, from Lemma 1.1, $\bigcap_{z \in L} E(z) \neq \emptyset$, i.e., there exists $y^* \in L$ such that

$$\Psi(z, -\eta(z, y^*)) \leq 0, \quad \forall z \in L.$$

It follows from (iv) that $y^* \in \Omega$ and $y^* \in H(z_j)$, $j = 1, 2, \ldots, m$. Therefore, H has the finite intersection property and so, $\bigcap_{z \in K} H(z) \neq \emptyset$. Then $S_{HS} = S_M \neq \emptyset$. Let $y_1^*, y_2^* \in S_M$ and $t \in (0, 1)$. Put $y_t = y_1 + t\eta(y_2, y_1) \in K$. Since for each $z \in K$, $\Psi(z, -\eta(z, \cdot))$ is invex with respect to η on K, one has

$$\begin{aligned}\Psi(z, -\eta(z, y_t)) &= \Psi(z, -\eta(z, y_1 + t\eta(y_2, y_1))) \\ &\leq t\Psi(z, -\eta(z, y_2)) + (1 - t)\Psi(z, -\eta(z, y_1)) \leq 0,\end{aligned}$$

i.e.,

$$\Psi(z, -\eta(z, y_t)) \leq 0, \quad \forall z \in K.$$

Consequently, S_M and S_{HS} are invex subsets of K with respect to η. To sum up, S_{HS} and S_M are two nonempty compact and invex sets with respect to η. Similarly, we can conclude that $\aleph_{HS}$ and $\aleph_M$ are two nonempty compact and invex sets with respect to η. This completes the proof. □

Remark 2.2 In Theorem 2.1, $K \subseteq R^n$ is an invex set with respect to η but not convex. We can not directly establish the existence of solutions for (HSBEP) and (MBEP) by using the Fan-KKM theorem. In view of this, we consider a convex subset $\mathcal{K}$ of R^n with $K \subseteq \mathcal{K}$.

Theorem 2.2 *Assume that all conditions of Theorem 2.1 are satisfied. Let $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in \mathcal{K}$ and Φ be strictly η-pseudomonotone on $\mathcal{K}$. Then $\aleph_{HS}$ and $\aleph_M$ are two singletons.*

Proof By Theorem 2.1, $\aleph_{HS}$ and $\aleph_M$ are two nonempty compact and invex sets with respect to η and so, $\aleph_{HS} = \aleph_M$. Let us show that $\aleph_{HS}$ and $\aleph_M$ are two singletons. Suppose to the contrary that there exist $x_1, x_2 \in \aleph_{HS}$ with $x_1 \neq x_2$. Then $x_1, x_2 \in S_{HS}$,

$$\Phi(x_1, \eta(y, x_1)) \geq 0, \quad \forall y \in S_{HS} \tag{2.6}$$

and

$$\Phi(x_2, \eta(y, x_2)) \geq 0, \quad \forall y \in S_{HS}. \tag{2.7}$$

Substituting $y = x_2$ and $y = x_1$ into (2.6) and (2.7), respectively, we have

$$\Phi(x_1, \eta(x_2, x_1)) \geq 0 \tag{2.8}$$

and

$$\Phi(x_2, \eta(x_1, x_2)) \geq 0. \tag{2.9}$$

Since Φ is strictly η-pseudomonotone on K, and from (2.9),

$$\Phi(x_1, -\eta(x_1, x_2)) < 0.$$

Owing to $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in \mathcal{K}$. Then $\eta(x_1, x_2) + \eta(x_2, x_1) = 0$, i.e., $\eta(x_2, x_1) = -\eta(x_1, x_2)$. Thus

$$\Phi(x_1, \eta(x_2, x_1)) < 0,$$

which contradicts (2.8). This completes the proof. □

In particular, if the lower invex equilibrium problems of (HSBEP) and (MBEP) have unique solutions, then (HSBEP) and (MBEP) also have unique solutions.

Theorem 2.3 *Assume that all conditions of Theorem 2.1 are satisfied. Let $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in \mathcal{K}$ and Ψ be strictly η-pseudomonotone on $\mathcal{K}$. Then $\aleph_{HS}$ and $\aleph_M$ are two singletons.*

Proof From Theorem 2.1, $\aleph_{HS}$ and $\aleph_M$ are nonempty. So, S_{HS} and S_M are nonempty. By the similar proof of that of Theorem 2.2, we can conclude that S_{HS} and S_M are two singletons. Without loss of generality, let $S_{HS} = S_M = \{y^*\}$. Since $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in \mathcal{K}$, one has $\eta(y^*, y^*) = 0$. Taking into account the positive homogeneity of Φ with respect to the second argument,

$$\Phi(y^*, \eta(y^*, y^*)) = \Phi(y^*, 0) = \Phi(y^*, \lambda\eta(y^*, y^*)) = \lambda\Phi(y^*, \eta(y^*, y^*)), \quad \forall\lambda > 0, \lambda \neq 1.$$

Moreover, one has $\Phi(y^*, \eta(y^*, y^*)) = 0$. Therefore, $\aleph_{HS} = \aleph_M = \{y^*\}$. This completes the proof. □

Example 2.1 Let $R^n = R = (-\infty, +\infty)$, $K = \mathcal{K} = [-1, 1]$, $\eta(x, y) = \frac{x-y}{2}$ for all $x, y \in \mathcal{K}$, and let $\Phi(x, \eta(y, x)) = \langle 2x, \frac{y-x}{2}\rangle = x(y - x)$ and $\Psi(y, \eta(z, y)) = \langle y, \frac{z-y}{2}\rangle = \frac{yz-y^2}{2}$ for all $x, y, z \in K$. Then K is invex with respect to η. Put $\Omega = \Xi = \{0\}$. It is easy to check that all conditions of Theorems 2.2 and 2.3 are satisfied. After computation, $\aleph_{HS} = \aleph_M = \{0\}$.

If $\eta(x, y) = x - y$ for all $x, y \in \mathcal{K}$, then, from Theorems 2.1 and 2.2, the following results hold:

Corollary 2.2 *Let K be a nonempty closed and convex subset of R^n, Φ and Ψ be positively homogeneous and continuous with respect to the second argument, and pseudomonotone, generalized subodd and hemicontinuous on $\mathcal{K}$. Assume that the following conditions hold:*

(i) For each $y, z \in K$, the functions $\Phi(y, \cdot - y)$ and $\Psi(z, \cdot - z)$ are convex on K;
(ii) There exists a nonempty closed bounded convex set $\Omega \subseteq K$ such that for each $y \in \mathcal{K}\backslash\Omega$, there exists $z \in \Omega$ that satisfies $\Psi(z, y - z) > 0$;
(iii) There exists a nonempty closed bounded convex set $\Xi \subseteq S_{HS}$ such that for each $x \in S_{HS}\backslash\Xi$, there exists $\tilde{y} \in \Xi$ that satisfies $\Phi(\tilde{y}, x - \tilde{y}) > 0$.

Then $\aleph_{HS}$ and $\aleph_M$ are nonempty compact and convex.

Corollary 2.3 *Assume that all conditions of Corollary 2.2 are satisfied. Let Φ be strictly pseudomonotone on $\mathcal{K}$. Then $\aleph_{HS}$ and $\aleph_M$ are two singletons.*

3 Applications

In this section, we shall apply the obtained results in Sect. 2 to solve bilevel pseudomonotone variational inequalities which was studied under the assumption of existence of its solution by Anh et al. [1] from the theory algorithm point of view, and study a class of minimization problem with variational inequality constraint.

(I) Bilevel pseudomonotone variational inequalities. Let C be a nonempty closed convex subset of R^n with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Anh et al. [1] studied the following so-called bilevel variational inequalities (BVI):

$$\text{find} \quad x^* \in Sol(G, C) \quad \text{such that} \quad \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in Sol(G, C), \tag{3.1}$$

where $F : Sol(G, C) \to R^n$, $Sol(G, C)$ is the solution set of the variational inequality:

$$\text{find} \quad y^* \in C \quad \text{such that} \quad \langle G(y^*), y - y^* \rangle \geq 0, \quad \forall y \in C, \tag{3.2}$$

where $G : C \to R^n$.

Let $\eta(z, y) = z - y$ for all $z, y \in R^n$. Then C is an invex subset of R^n with respect to the η.

Definition 3.1 [1] Let C be a nonempty subset of R^n, and $A : C \to R^n$ be a vector-valued mapping. A is said to be

(i) monotone if $\langle Ay - Az, y - z \rangle \geq 0$ for all $y, z \in C$;
(ii) pseudomonotone if for any $y, z \in C$,

$$\langle A(y), z - y \rangle \geq 0 \Rightarrow \langle Az, z - y \rangle \geq 0;$$

(iii) strictly pseudomonotone on C if for any $y, z \in C$,

$$\langle A(y), z - y \rangle \geq 0 \Rightarrow \langle Az, z - y \rangle > 0$$

and A is pseudomonotone on C.

Remark 3.1 Let $\eta(z, y) = z - y$ and $\Psi(y, z - y) = \langle A(y), z - y \rangle$ for all $z, y \in C$. It is easy to check that the following statements are true:

(i) if A is pseudomonotone on C, then Ψ is also η-pseudomonotone on C;
(ii) if A is strictly pseudomonotone on C, then Ψ is also strictly η-pseudomonotone on C.

Theorem 3.1 *Let C be a nonempty closed and convex subset of R^n, G and F be continuous and pseudomonotone on C. Assume that there exists a nonempty closed bounded convex set $\Delta \subseteq C$ such that for each $y \in C\backslash\Delta$, there exists $z \in \Delta$ that satisfies $\langle G(z), y - z\rangle > 0$. If there exists a nonempty closed bounded convex set $\nabla \subseteq Sol(G, C)$ such that for each $x \in Sol(G, C)\backslash\nabla$, there exists $\tilde{y} \in \nabla$ that satisfies $\langle F(\tilde{y}), x - \tilde{y}\rangle > 0$, then the solution set of (BVI) is nonempty compact and convex.*

Proof Let $\eta(y, x) = y - x$, $\Phi(x, y - x) = \langle F(x), y - x\rangle$ and $\Psi(y, z - y) = \langle G(y), z - y\rangle$ for all $x, y, z \in C$. For each $y, z \in C$ and $t \in (0, 1)$, $\eta(y + t\eta(z, y), y) = y + t(z - y) - y = t(z - y) = t\eta(z, y)$, and moreover, condition (i) of Theorem 2.1 is satisfied. It is easy to see that conditions (iv) and (v) of Theorem 2.1 are also true. We only need to prove that the conditions (ii) and (iii) of Theorem 2.1 hold. By the definition of Φ and Ψ, Φ and Ψ are positively homogeneous and continuous with respect to the second argument. Since G and F be continuous and pseudomonotone on C, from Remark 3.1, it follows that Φ and Ψ are η-pseudomonotone and η-hemicontinuous on C. For any $x \in C$, $d_j \in R^n (j = 1, 2, \ldots, m)$ and $\sum_{j=1}^{m} d_j = 0$,

$$\sum_{j=1}^{m} \Phi(x, d_j) = \sum_{j=1}^{m} \langle F(x), d_j\rangle = \left\langle F(x), \sum_{j=1}^{m} d_j \right\rangle \geq 0$$

and

$$\sum_{j=1}^{m} \Psi(x, d_j) = \sum_{j=1}^{m} \langle G(x), d_j\rangle = \left\langle G(x), \sum_{j=1}^{m} d_j \right\rangle \geq 0.$$

That is, Φ and Ψ are generalized subodd. Note that for each $y, z \in C$, $\Phi(y, -\eta(y, \cdot)) = \langle F(y), \cdot - y\rangle$ and $\Psi(z, -\eta(z, \cdot)) = \langle G(z), \cdot - z\rangle$. Then, for each $y, z \in K$, the functions $\Phi(y, -\eta(y, \cdot))$ and $\Psi(z, -\eta(z, \cdot))$ are convex on C, i.e., $\Phi(y, -\eta(y, \cdot))$ and $\Psi(z, -\eta(z, \cdot))$ is invex with respect to the η on C. To sum up, all conditions of Theorem 2.1 hold. Therefore, from Theorem 2.1, the solution set of (BVI) is nonempty compact and convex. This completes the proof. □

Theorem 3.2 *Assume that all conditions of Theorem 3.1 are satisfied. Let F be strictly pseudomonotone on C. Then the solution set of (BVI) is a singleton.*

Proof It immediately follows from Theorems 2.2 and 3.1 and Remark 3.1 (ii). This completes the proof

(II) Minimization problem with variational inequality constraint Let D be a nonempty closed and invex subset of $R = (-\infty, +\infty)$ with respect to the mapping $\eta : R \times R \to R$, $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in D$, $\mathcal{D}$ be a closed convex subset of R with $D \subseteq \mathcal{D}$ and let the functions $f : D \to R$ and $g : D \to R$ be continuously differentiable. Further, we assume that both the functions f and g are invex with respect to the same $\eta : R \times R \to R$. Now we consider the following minimization problem with variational inequality constraint (MPEC):

$$\min_{x \in S_g} \quad f(x),$$

where S_g is the solution set of the variational inequality:

$$\text{find} \quad x \in D \quad \text{such that} \quad \langle g(x), \eta(y, x) \rangle \geq 0, \quad y \in D.$$

Definition 3.2 [3] Let $\eta : R \times R \to R$. g is said to be

(i) monotone with respect to η if, $\langle g(y) - g(x), \eta(y, x) \rangle \geq 0$ for all $x, y \in D$;
(ii) strictly monotone with respect to η if, $\langle g(y) - g(x), \eta(y, x) \rangle > 0$ for all $x, y \in D$.

Lemma 3.1 [27,28] *Let f be a differentable function on the invex subset K of R and η be a bifunction such that $\eta(x, x + t\eta(y, x)) = -t\eta(y, x)$ and $\eta(y, x + t\eta(y, x)) = (1 - t)\eta(y, x)$ for all $x, y \in K$ and $t \in [0, 1]$. Then the following are equivalent:*

(i) the function f is invex with respect to η;
(ii) $f(y_2) - f(y_1) \geq \langle f'(y_1), \eta(y_2, y_1) \rangle$ for all $y_1, y_2 \in K$, where $f'(y_1)$ is the differential of f at y_1;
(iii) f' is monotone with respect to η.

It is easy to obtain the following result from Lemma 3.1.

Lemma 3.2 *Assume that all conditions of Lemma 3.1 are satisfied. If f and $-f$ are invex with respect to η, then $x \in K$ is a solution of the problem $\min_{x \in K} f(x)$ if and only if it is a solution of the following variational inequality:*

$$\textit{find} \quad x \in K \quad \textit{such that} \quad \langle f'(x), \eta(y, x) \rangle \geq 0, \quad \forall y \in K.$$

Now we study the existence of solution of (MPEC) by using Theorem 2.1.

Theorem 3.3 *Let $\eta : R \times R \to R$ be affine with respect to the first argument, and continuous with respect to the second argument and $D \subseteq R$ be a nonempty closed and invex set with respect to η with $\eta(y, y + t\eta(z, y)) = -t\eta(z, y)$, $\eta(z, y + t\eta(z, y)) = (1 - t)\eta(z, y)$ and $\eta(y + t\eta(z, y), y) = t\eta(z, y)$ for all $y, z \in D$ and $t \in (0, 1)$. Assume that the following conditions hold:*

(i) f and $-f$ are invex with respect to η;
(ii) For each $y, z \in D$, the functions $\langle f'(y), -\eta(y, \cdot) \rangle$ and $\langle g(z), -\eta(z, y) \rangle$ are invex with respect to the η on D;
(iii) There exists a nonempty closed bounded convex set $W \subseteq D$ such that for each $y \in \mathcal{D} \backslash W$, there exists $z \in W$ that satisfies that $\langle g(z), -\eta(z, \cdot) \rangle > 0$;
(iv) There exists a nonempty closed bounded convex set $T \subseteq S_g$ such that for each $x \in S_g \backslash T$, there exists $\tilde{y} \in T$ that satisfies $\langle f'(\tilde{y}), -\eta(\tilde{y}, x) \rangle > 0$.

Then (MPEC) is solvable.

Proof Let $\Phi(x, \eta(y, x)) = \langle f'(x), \eta(y, x) \rangle$ and $\Psi(y, \eta(z, y)) = \langle g(y), \eta(z, y) \rangle$ for all $x, y, z \in D$. Note that η is continuous with respect to the second argument, and $\eta(y, y + t\eta(z, y)) = -t\eta(z, y)$ for all $y, z \in D$ and $t \in (0, 1)$. From these, we can derive that $\eta(y, y) = 0$ for all $y \in D$. Clearly, Φ and Ψ satisfy the conditions (ii)–(iv) of Theorem 2.1 By the same argument of Theorem 2.1, we have that S_g is nonempty compact and invex with respect to η. Since f and $-f$ are invex with respect to η, from

Lemma 3.2, we know that $x \in S_g$ is a solution of the problem $\min_{x \in S_g} f(x)$ if and only if it is a solution of the variational inequality (VI): find $x \in S_g$ such that

$$\Phi(x, \eta(y, x)) = \langle f'(x), \eta(y, x) \rangle \geq 0, \quad \forall y \in S_g.$$

That is, (MPEC) is equivalent to the following bilevel equilibrium problem: find $x \in S_g$ such that

$$\Phi(x, \eta(y, x)) = \langle f'(x), \eta(y, x) \rangle \geq 0, \quad \forall y \in S_g, \tag{3.3}$$

where S_g is the solution set of the equilibrium as follows: find $y \in D$ such that

$$\Psi(y, \eta(y, z)) = \langle g(y), \eta(z, y) \rangle \geq 0, \quad \forall z \in D. \tag{3.4}$$

Clearly, Ψ satisfies all conditions of Theorem 2.1 It follows from Theorem 2.1 that the bilevel equilibrium problem (3.3) with (3.4) has a solution. Therefore, (MPEC) is solvable. This completes the proof. □

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