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Research Article

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L^q-regularization for the inverse Robin problem

Abstract: In this paper, an inverse Robin problem for an elliptic equation is investigated. The Robin coefficient is assumed to be a small perturbation from a reference coefficient, hence a sparse structure is a priori known. An L^q-regularized output least square formulation is considered with $q \in (0,1)$. The existence of the minimizer is proved. The convergence of the finite element discretization with respect to mesh size is shown. A modified Newton algorithm is applied to solve the smoothing first order optimality system. Two-dimensional and three-dimensional numerical examples demonstrate the efficiency of the proposed method.

Keywords: Inverse Robin problem, L^q-regularization, Newton method, finite element method

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1 Introduction

In this paper, we consider the inversion of the Robin coefficient γ in the following elliptic equation with mixed boundary conditions:

$$\begin{cases} -\nabla \cdot (\alpha \nabla u) = f & \text{in } \Omega, \\ \alpha \dfrac{\partial u}{\partial n} = v & \text{on } \Gamma_c, \\ \alpha \dfrac{\partial u}{\partial n} + \gamma u = \gamma u_a & \text{on } \Gamma_i, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is an open bounded polyhedral domain with boundary $\Gamma = \Gamma_c \cup \Gamma_i$. This equation has a lot of applications. For example, in thermal engineering, it describes steady-state heat transfer with convective conduction occurring at the interface Γ_i between the conducting body and the ambient environment, where α, f, v, u_a refer to the heat conductivity, the heat source, the heat flux and the ambient temperature, respectively. The disjointed boundaries Γ_c and Γ_i refer to the experimentally accessible and inaccessible parts, respectively. The Robin coefficient γ is closely related to the physical and chemical property of the material on the boundary. In some applications such as corrosion detection [9] and nondestructive detection of metal-oxide-silicon field effect transistor (MOSFET) [6], one needs to obtain the information of the material on the inaccessible boundary Γ_i which naturally leads to the inverse problem of estimating the Robin coefficient γ on the boundary Γ_i from the Cauchy data

$$u = g^\delta \quad \text{and} \quad \alpha \frac{\partial u}{\partial n} = v \quad \text{on } \Gamma_c$$

measured on the boundary Γ_c. It is known that the Cauchy problem is severely ill-posed in the sense that the solution does not depend on the data continuously [2]. Several regularization methods have been proposed to overcome this difficulties based on different a priori information [3, 7, 9, 13–15].

In practical scenarios such as corrosion detection the defects may only occurs at small spots of the device, while in MOSFET [6] the imperfect contact usually occurs only locally and thus the difference between the profile to be identified and the reference profile may have a small support. Hence it is natural to consider the inverse Robin problem with such a priori information. The investigation of regularization schemes with sparsity promoting regularizer has been one of the hottest topics in the field of inverse problems over the last ten years, see [12] for an overview. A sparsity promoted L^1-regularization formulation with box constraints for inverse Robin problem is considered in [16]. However if the noise is very large or the support of the desired function is very small, the L^1-regularization may not have a satisfactory result. On the other hand, it is generally believed that ℓ^q- and L^q-regularization ($q \in [0, 1)$) approach may capture the sparse structure better than ℓ^1 or L^1-regularization, see [4, 11].

Let $\gamma^\dagger$ and $\widehat{\gamma}$ be the reference Robin coefficient and the coefficient to be recovered, respectively. It is reasonable to assume that $\gamma^\dagger - \widehat{\gamma}$ has a small support and exhibits a sparse structure. To recover $\widehat{\gamma}$, we consider the following L^q-regularized functional to handle possible large noise and small support:

$$J_{\lambda,\beta}(\gamma) = \frac{1}{2}\|u(\gamma) - g^\delta\|^2_{L^2(\Gamma_c)} + \lambda\|\gamma - \gamma^\dagger\|_{L^q(\Gamma_i)} + \frac{\beta}{2}|\gamma|^2_{H^1(\Gamma_i)},$$

where $u(\gamma)$ is the solution of (1.1) for a given γ. The first fidelity term in $J_{\lambda,\beta}$ fits the observation data g^δ, the second L^q-regularization term enforces the sparsity, and the last H^1-term stabilizes the formulation whereas the regularization parameter $\lambda, \beta > 0$ tradeoff these three terms. To ensure the well-posedness of $u(\gamma)$ we assume that $\gamma \in \mathcal{A}$, where

$$\mathcal{A} = \{\gamma \in H^1(\Gamma_i) : 0 < \underline{\gamma} \le \gamma(x) \le \overline{\gamma} < +\infty \text{ for all } x \in \Gamma_i\}.$$

Now the Robin coefficient is recovered by:

Problem 1.1. Consider

$$\min_{\gamma\in\mathcal{A}} J_{\lambda,\beta}(\gamma).$$

Due to the nonconvex and nonsmooth structure of the L^q-norm, to find the efficient numerical algorithm is nontrivial. The iteratively reweighted least squares method [4, 11] and the iterative thresholding algorithm were applied to solve the corresponding optimization problem, see [12] for an overview. In this article we consider the smoothed version of the L^q-norm and adopt a Newton type algorithm to solve its first order necessary Karush–Kuhn–Tucker system.

The rest of this paper is organized as follows. In Section 2 we study the existence and stability of the minimizers of the L^q-regularized minimization problem and its smoothed version. Finite element discretization is given in Section 3, and the convergence of the finite element solution is proved. In Section 4 a modified Newton algorithm is proposed to solve the smoothed minimization problem. Numerical experiments for the two- and three-dimensional cases are given in Section 5 to verify the efficiency of our approach.

We end this section with some notations and assumptions. For a given real number r, let H^r be the standard Sobolev space endowed with the norm $\|\cdot\|_{H^r}$ and the seminorm $|\cdot|_{H^r}$, see [1] for details. By $L^2(\Gamma_c)$ and $L^2(\Gamma_i)$ we denote the standard square integrable function spaces endowed with the norm $\|\cdot\|_{L^2(\Gamma_c)}$ and $\|\cdot\|_{L^2(\Gamma_i)}$ respectively. For $q \in (0, 1)$ we define $L^q(\Gamma_i)$ as the complete metric space of measurable functions $\gamma(s)$ satisfying

$$\|\gamma\|_{L^q(\Gamma_i)} := \int_{\Gamma_i} |\gamma(s)|^q \, ds < +\infty.$$

We assume $\alpha \in H^1(\Omega)$ with $0 < \underline{c} \le \alpha(x) \le \bar{c} < \infty$, $f \in (H^1(\Omega))'$, $g^\delta \in H^{\frac{1}{2}}(\Omega)$, $v \in (H^{\frac{1}{2}}(\Omega))'$.

2 L^q-regularization

We first prove the existence of a minimizer of Problem 1.1. Since the L^q-space lacks of weak lower semicontinuity properties, the existence is not straightforward. We will follow the idea as in [11].

Theorem 2.1. *For any $\beta, \lambda > 0$, there exists at least one minimizer γ^* to Problem 1.1.*

Proof. It is known that $\mathcal{A}$ is nonempty and $J_{\lambda,\beta}(\gamma)$ is bounded from below, and thus there exists a minimum sequences $\{\gamma^k\}_k$ such that

$$\lim_{k\to+\infty} J_{\lambda,\beta}(\gamma^k) = \inf_{\gamma} J_{\lambda,\beta}(\gamma).$$

Then $\{|\gamma^k|_{H^1(\Gamma_i)}\}_k$ and $\{\|\gamma^k\|_{L^q(\Gamma_i)}\}_k$ are bounded sequences. Decompose $\gamma^k = \gamma_0^k + \gamma_1^k$, where

$$\gamma_0^k = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} \gamma^k(s)\,ds \quad \text{and} \quad \gamma_1^k = \gamma^k - \gamma_0^k.$$

Observing the fact that

$$\frac{1}{|\Gamma_i|}\int_{\Gamma_i} \gamma_1^k(s)\,ds = 0,$$

then $\{\|\gamma_1^k\|_{H^1(\Gamma_i)}\}_k$ is bounded by Poincaré inequality. Therefore $\{\|\gamma_1^k\|_{L^q(\Gamma_i)}\}_k$ is bounded, and hence

$$\|\gamma_0^k\|_{L^q(\Gamma_i)} \le \|\gamma^k\|_{L^q(\Gamma_i)} + \|\gamma_1^k\|_{L^q(\Gamma_i)}$$

is bounded. Since γ_0^k is a constant function, we have $\{\|\gamma^k\|_{H^1(\Gamma_i)}\}_k$ is bounded. We pass to a subsequence which is still denoted by $\{\gamma^k\}_k$. There exists some $\gamma^* \in H^1(\Gamma_i) \cap \mathcal{A}$ such that $\gamma^k \to \gamma^*$ weakly in $H^1(\Gamma_i)$. Then

$$\|\gamma^k - \gamma^*\|_{L^2(\Gamma_i)} \to 0 \tag{2.1}$$

follows from the compact embedding from $H^1(\Gamma_i)$ into $L^2(\Gamma_i)$. By [14, Theorem 3.2], we get

$$\|u(\gamma^k) - u(\gamma^*)\|_{L^2(\Gamma_c)} \to 0. \tag{2.2}$$

By the Hölder inequality we obtain that

$$\|\gamma^k - \gamma^\dagger\|_{L^q(\Gamma_i)} \to \|\gamma^* - \gamma^\dagger\|_{L^q(\Gamma_i)}. \tag{2.3}$$

Combining (2.2)–(2.3) with the weak lower semicontinuity of the H^1-seminorm, we obtain that γ^* is a minimizer of Problem 1.1. □

Remark 2.2. In this article we focus on the minimization Problem 1.1 with fixed parameters λ and β. But the choice of proper regularization parameters λ and β is also important in both regularization theory and numerical realization. Let the true Robin coefficient be $\widehat{\gamma} \in \mathcal{A}$ and the noise level be $\delta = \|u(\widehat{\gamma}) - g^\delta\|_{L^2(\Gamma_c)}$. By the standard Tikhonov regularization theory (e.g. [5]), we can prove that if the regularization parameters are chosen properly in the sense

$$\frac{\lambda(\delta)}{\beta(\delta)} = \text{constant} \quad \text{and} \quad \lim_{\delta\to 0^+} \max\Big\{\beta(\delta), \frac{\delta^2}{\beta(\delta)}\Big\} = 0, \tag{2.4}$$

then the minimizer $\gamma^*_{\lambda(\delta),\beta(\delta)}$ of Problem 1.1 converges to $\widehat{\gamma}$ in $L^2(\Gamma_i)$ when noise level $\delta \to 0^+$. The proof is rather standard. Since

$$J_{\lambda(\delta),\beta(\delta)}(\gamma^*_{\lambda(\delta),\beta(\delta)}) \le J_{\lambda(\delta),\beta(\delta)}(\widehat{\gamma}) = \frac{\delta^2}{2} + \lambda(\delta)\|\widehat{\gamma}\|_{L^q(\Gamma_i)} + \frac{\beta(\delta)}{2}|\widehat{\gamma}|^2_{H^1(\Gamma_i)} \tag{2.5}$$

and assumption (2.4), by a similar argument as in Theorem 2.1 we have $\{\gamma^*_{\lambda(\delta),\beta(\delta)}\}_\delta$ is bounded in $L^q(\Gamma_i)$ and $H^1(\Gamma_i)$. Hence there exist a subsequence $\{\gamma^*_{\lambda(\delta_k),\beta(\delta_k)}\}_{\delta_k}$ and $\widetilde{\gamma}$ such that $\gamma^*_{\lambda(\delta_k),\beta(\delta_k)}$ weakly converge to $\widetilde{\gamma}$ in $H^1(\Gamma_i)$. From [14, Theorem 3.2], (2.5) and (2.4) we have

$$\begin{aligned}
\|u(\widetilde{\gamma}) - u(\widehat{\gamma})\|_{L^2(\Gamma_c)} &= \lim_{\delta_k\to 0^+} \|u(\gamma^*_{\lambda(\delta_k),\beta(\delta_k)}) - u(\widehat{\gamma})\| \\
&\le \lim_{\delta_k\to 0^+} (\delta_k + \|u(\gamma^*_{\lambda(\delta_k),\beta(\delta_k)}) - g^{\delta_k}\|) \\
&\le \lim_{\delta_k\to 0^+} \sqrt{2J_{\lambda(\delta_k),\beta(\delta_k)}(\gamma^*_{\lambda(\delta_k),\beta(\delta_k)})} = 0.
\end{aligned}$$

Due to the uniqueness of the inverse Robin problem, which can be obtained by the unique continuation argument, see [10], we deduce that $\widetilde{\gamma} = \widehat{\gamma}$. Moreover this uniqueness implies the convergence for the sequence, and hence

$$\lim_{\delta\to 0^+} \|\gamma^*_{\lambda(\delta),\beta(\delta)} - \widehat{\gamma}\|_{L^2(\Gamma_i)} = 0.$$

By applying similar argument as in [14, Theorem 3.3] and (2.3), one can show that the minimizer of Problem 1.1 continuously depends on the observing data g^δ.

Theorem 2.3. *Let γ^k be a minimizer of Problem 1.1 with g^δ replaced by g^k. If $g^k \to g^\delta$ in $L^2(\Gamma_c)$, then there exists a subsequence of γ^k converging to γ^* in $L^2(\Gamma_i)$.*

To overcome the numerical difficult of L^q-penalty, we consider the following smooth version of Problem 1.1.

Problem 2.4. Consider

$$\min_{\gamma \in \mathcal{A}} J^\epsilon_{\lambda,\beta}(\gamma) = \frac{1}{2}\|u(\gamma) - g^\delta\|^2_{L^2(\Gamma_c)} + \lambda \Big\|\sqrt{(\gamma - \gamma^\dagger)^2 + \epsilon^2}\Big\|_{L^q(\Gamma_i)} + \frac{\beta}{2}|\gamma|^2_{H^1(\Gamma_i)},$$

where $\epsilon > 0$ is a small given parameter.

Theorem 2.5. *For any $\beta, \lambda, \epsilon > 0$, there exists at least one minimizer γ^ϵ to Problem 2.4.*

Proof. If (2.1) holds, then by simple computation we have

$$\Big\|\sqrt{(\gamma^k - \gamma^\dagger)^2 + \epsilon^2}\Big\|_{L^q(\Gamma_i)} \to \Big\|\sqrt{(\gamma^* - \gamma^\dagger)^2 + \epsilon^2}\Big\|_{L^q(\Gamma_i)}. \tag{2.6}$$

Therefore the proof is similar to the proof of Theorem 2.1 and we omit it here. □

The next theorem shows that the solution to Problem 2.4 converges to the solution to Problem 1.1 as ϵ goes to 0.

Theorem 2.6. *Let $\epsilon_k \to 0^+$ and γ^k be a minimizer of $J^{\epsilon_k}_{\lambda,\beta}$. Then the sequence $\{\gamma^k\}_k$ contains a subsequence that converges in $L^2(\Gamma_i)$ to a minimizer of Problem 1.1.*

Proof. A similar argument as the proof of Theorem 2.1 shows that $\{\gamma^k\}_k$ contains a subsequence (still denoted by $\{\gamma^k\}_k$) converges to some γ^* such that (2.1) and

$$J_{\lambda,\beta}(\gamma^*) \le \liminf_{k\to\infty} J_{\lambda,\beta}(\gamma^k)$$

hold. Then we need to show that γ^* is the minimizer of Problem 1.1. This part is similar as [16, Theorem 3.3], we include here the main steps for completeness. For any given $\gamma \in \mathcal{A}$, by the inequality

$$\int_{\Gamma_i} (|\gamma - \gamma^\dagger|^2 + \epsilon_k^2)^{q/2} \le \int_{\Gamma_i} (|\gamma - \gamma^\dagger| + \epsilon_k)^q \le \int_{\Gamma_i} (|\gamma - \gamma^\dagger|^q + \epsilon_k^q)$$

we have

$$J_{\lambda,\beta}(\gamma) \le J^{\epsilon_k}_{\lambda,\beta}(\gamma) \le J_{\lambda,\beta}(\gamma) + \lambda\epsilon_k^q|\Gamma_i|.$$

Therefore, for any $\gamma \in \mathcal{A}$, there holds

$$J_{\lambda,\beta}(\gamma) + \lambda\epsilon_k^q|\Gamma_i| \ge J^{\epsilon_k}_{\lambda,\beta}(\gamma) \ge J^{\epsilon_k}_{\lambda,\beta}(\gamma^k) \ge J_{\lambda,\beta}(\gamma^k).$$

Taking limit for $k \to \infty$, we conclude that γ^* is a minimizer of Problem 1.1. □

3 Finite element approximation

In this section we consider the finite element approximation for Problem 1.1 and Problem 2.4. Since the argument and results are very similar for these two cases, we will focus on the latter one. A quasi-uniform triangulation of Ω is denoted by T_h, the corresponding finite element space V_h and discrete admissible set $\mathcal{A}_h$ are defined by

$$V_h = \{v_h \in C(\bar{\Omega}) : v_h|_\Delta \in P_1(\Delta) \text{ for all } \Delta \in T_h\}, \quad \mathcal{A}_h = \mathcal{A} \cap V_{h,\Gamma_i},$$

where V_{h,Γ_i} is the restriction of V_h on Γ_i. Then Problem 2.4 can be approximated as follows.

Problem 3.1. Consider

$$\min_{\gamma_h\in\mathcal{A}_h} J^{\epsilon,h}_{\lambda,\beta}(\gamma_h)=\frac{1}{2}\|u_h(\gamma_h)-g^\delta\|^2_{L^2(\Gamma_c)}+\lambda\Big\|\sqrt{(\gamma_h-\gamma^\dagger)^2+\epsilon^2}\Big\|_{L^q(\Gamma_i)}+\frac{\beta}{2}|\gamma_h|^2_{H^1(\Gamma_i)},$$

where $u_h(\gamma_h)$ is the discrete solution of (1.1), i.e., for all $\phi_h\in V_h$ we have

$$\int_\Omega \alpha\nabla u_h(\gamma_h)\cdot\nabla\phi_h+\int_{\Gamma_i}\gamma_h u_h(\gamma_h)\phi_h=\int_{\Gamma_i}\gamma_h u_a\,\phi_h+\int_{\Gamma_c}\upsilon\,\phi_h+\int_\Omega f\phi_h. \tag{3.1}$$

Note that Problem 3.1 is a finite-dimensional optimization problem. By virtue of the continuous and coercivity of the discrete functional, the existence of a minimizer to Problem 3.1 is straightforward.

Theorem 3.2. *For any $\beta,\lambda,\epsilon>0$, there exists at least one minimizer γ_h to Problem 3.1.*

Next we show the convergence of the minimizers of the discrete optimization Problem 3.1 as the mesh size h goes to 0.

Theorem 3.3. *Let $\{\gamma_{h_k}\}_{h_k}$ be a sequences of minimizers of Problem 3.1 as $h_k\to 0$. Then there is a subsequence still denoted by $\{\gamma_{h_k}\}_{h_k}$ such that γ_{h_k} converges strongly to γ^*, a minimizer of Problem 2.4, in $L^2(\Gamma_i)$.*

Proof. For simplicity $u_{h_k}(\gamma_{h_k})$ is denoted by u_{h_k}. By a similar argument as in Theorem 2.1, $\{\gamma_{h_k}\}$ is bounded in $H^1(\Gamma_i)$. Let $\phi_h=u_{h_k}$ in equation (3.1). We find $\{u_{h_k}\}$ is also bounded in $H^1(\Omega)$. By passing to a subsequence which is still denoted by h_k, we have

$$\gamma_{h_k}\to\gamma^*\ \text{ weakly in } H^1(\Gamma_i),\qquad u_{h_k}\to u^*\ \text{ weakly in } H^1(\Omega).$$

For any $\phi\in C^\infty(\Omega)$, we choose $\phi_{h_k}\in V_{h_k}$ such that ϕ_{h_k} converges to ϕ strongly in $H^1(\Omega)$. Taking limit in equation (3.1), noticing the weak-strong convergence for first term and strong convergence for others, together with a density argument, we obtain $u^*=u(\gamma^*)$. Then a similar argument as for Theorem 2.5 implies that

$$\Big\|\sqrt{(\gamma_{h_k}-\gamma^\dagger)^2+\epsilon^2}\Big\|_{L^q(\Gamma_i)}\to\Big\|\sqrt{(\gamma^*-\gamma^\dagger)^2+\epsilon^2}\Big\|_{L^q(\Gamma_i)},$$

and hence

$$J^\epsilon_{\lambda,\beta}(\gamma^*)\le\liminf_k J^{\epsilon,h}_{\lambda,\beta}(\gamma_{h_k}).$$

Next we show that γ^* is a minimizer of Problem 2.4. For any $\gamma\in\mathcal{A}\cap C^1(\Gamma_i)$, let $\mathcal{I}_{h_k}\gamma$ be its interpolation to $\mathcal{A}_{h_k}$. Then by the standard approximation property of finite element we have $\|\mathcal{I}_{h_k}\gamma-\gamma\|_{H^1(\Gamma_i)}\to 0$, and hence $u_{h_k}(\mathcal{I}_{h_k}\gamma)\to u(\gamma)$ weakly in $H^1(\Omega)$. It follows

$$J^\epsilon_{\lambda,\beta}(\gamma)=\lim_k J^{\epsilon,h}_{\lambda,\beta}(\mathcal{I}_{h_k}\gamma)\ge\liminf_k J^{\epsilon,h}_{\lambda,\beta}(\gamma_{h_k})\ge J^\epsilon_{\lambda,\beta}(\gamma^*),$$

which implies that γ^* is a minimizer of Problem 2.4. □

4 KKT condition and a modified Newton algorithm

In this section we first derive the necessary optimal condition of the finite element approximation Problem 3.1. Then we adopt a modified Newton method to solve the KKT condition.

Theorem 4.1. *Let γ_h be a minimizer of Problem 2.4 such that the box constraint $\underline{\gamma}\le\gamma_h\le\overline{\gamma}$ never be active. Then there exist u_h, p_h such that (γ_h,u_h,p_h) satisfies the first-order necessary Karush–Kuhn–Tucker condition:*

$$\begin{cases}\displaystyle\int_\Omega\alpha\nabla u_h\cdot\nabla\phi_h+\int_{\Gamma_i}\gamma_h u_h\phi_h=\int_{\Gamma_i}\gamma_h u_a\phi_h+\int_{\Gamma_c}\upsilon\phi_h+\int_\Omega f\phi_h & \text{for all } \phi_h\in V_h,\\[2mm] \displaystyle\int_\Omega\alpha\nabla p_h\cdot\nabla\phi_h+\int_{\Gamma_i}\gamma_h p_h\phi_h=\int_{\Gamma_c}(u_h-g^\delta)\phi_h & \text{for all } \phi_h\in V_h,\\[2mm] \displaystyle\beta\int_{\Gamma_i}\nabla\gamma_h\cdot\nabla\phi_h+\int_{\Gamma_i}(u_h-u_a)p_h\phi_h+\int_{\Gamma_i}\frac{q\lambda(\gamma_h-\gamma^\dagger)\phi_h}{((\gamma_h-\gamma^\dagger)^2+\epsilon^2)^{1-\frac{q}{2}}}=0 & \text{for all } \phi_h\in V_{h,\Gamma_i}.\end{cases} \tag{4.1}$$

Proof. Analogue to the continuous case in [14, Theorem 3.5], we can show $\frac{1}{2}\|u_h(\gamma_h) - g^\delta\|^2_{L^2(\Gamma_c)}$ is Fréchet differentiable and its derivative is given by $(u_a - u_h)p_h|_{\Gamma_i}$, where u_h and p_h satisfies the first two equations in (4.1). By direct computation, one can show that the derivative with respect to γ_h implies the last equation. □

Remark 4.2. If the box constraint is active, we can introduce a Lagrange multiplier corresponding to it, see [16] for details. In this work we neglect the constraint since it is not our major concern. Moreover we can always project the numerical solution back onto $\mathcal{A}_h$ after each iteration.

The straightforward application of the Newton algorithm to solve the nonlinear KKT system (4.1) may cause numerical instability. Indeed, the derivative of the third equation in (4.1) has one term:

$$q(q-1)((\gamma_h - \gamma^\dagger)^2 + \epsilon^2)^{\frac{q}{2}-1} + q(2-q)\epsilon^2((\gamma_h - \gamma^\dagger)^2 + \epsilon^2)^{\frac{q}{2}-2}.$$

If $(\gamma_h - \gamma^\dagger)$ is away from 0 and $\epsilon = o(1)$, this term is negative (due to $q(q-1) < 0$), which shifts the spectral of $-\beta\Delta$ to left and may cause trouble during Newton iteration. In order to overcome this trouble, we proposed a modified Newton method. To be precise, we replace

$$\frac{1}{((\gamma_h - \gamma^\dagger)^2 + \epsilon^2)^{1-\frac{q}{2}}}$$

in the third equation of KKT system (4.1) by

$$\frac{1}{((\gamma_h^k - \gamma^\dagger)^2 + \epsilon^2)^{1-\frac{q}{2}}},$$

where γ_h^k is from the previous iteration. This approach can be viewed as the combination of the Newton method with the iteratively reweighted least square method. The details of algorithm can be found in Algorithm 1. The stop condition in line 6 can be chosen as the relative error

$$\max\left\{\frac{\|\gamma_h^{k+1} - \gamma_h^k\|_{L^2(\Gamma_i)}}{\|\gamma_h^{k+1}\|_{L^2(\Gamma_i)}}, \frac{\|u_h^{k+1} - u_h^k\|_{L^2(\Omega)}}{\|u_h^{k+1}\|_{L^2(\Omega)}}, \frac{\|p_h^{k+1} - p_h^k\|_{L^2(\Omega)}}{\|p_h^{k+1}\|_{L^2(\Omega)}}\right\}$$

is less than a given tolerance τ or the max iteration number exceeds a given value, i.e., $k + 1 \geq K_{\max}$.

Algorithm 1. Modified Newton algorithm.

1: Given initial guess γ_h^0, u_h^0, p_h^0.
2: **for** $k = 0, 1, 2, \dots$ **do**
3: Let $\omega_h^k := [(\gamma_h^k - \gamma^\dagger)^2 + \epsilon^2]^{\frac{q}{2}-1}$.
4: Obtain $\gamma_h^{k+1}, u_h^{k+1}, p_h^{k+1}$ by solving

$$\begin{cases} \int_\Omega \alpha\nabla u_h^{k+1}\cdot\nabla\phi_h + \int_{\Gamma_i}\gamma_h^k u_h^{k+1}\phi_h + \int_{\Gamma_i}\gamma_h^{k+1}u_h^k\phi_h - \int_{\Gamma_i}\gamma_h^k u_h^k\phi_h = \int_{\Gamma_i}\gamma_h^{k+1}u_a\phi_h + \int_{\Gamma_c}v\phi_h + \int_\Omega f\phi_h & \text{for all } \phi_h \in V_h, \\ \int_\Omega \alpha\nabla p_h^{k+1}\cdot\nabla\phi_h + \int_{\Gamma_i}\gamma_h^k p_h^{k+1}\phi_h + \int_{\Gamma_i}\gamma_h^{k+1}p_h^k\phi_h - \int_{\Gamma_i}\gamma_h^k p_h^k\phi_h = \int_{\Gamma_c}(u_h^{k+1} - g^\delta)\phi_h & \text{for all } \phi_h \in V_h, \\ \beta\int_{\Gamma_i}\nabla\gamma_h^{k+1}\cdot\nabla\phi_h + \int_{\Gamma_i}(u_h^{k+1} - u_a)p_h^k\phi_h + \int_{\Gamma_i}(u_h^k - u_a)p_h^{k+1}\phi_h & \\ \qquad - \int_{\Gamma_i}(u_h^k - u_a)p_h^k\phi_h + \int_{\Gamma_i}q\lambda(\gamma_h^{k+1} - \gamma^\dagger)\omega_h^k\phi_h = 0 & \text{for all } \phi_h \in V_{h,\Gamma_i}. \end{cases}$$

5: Projection γ_h^{k+1} onto $\mathcal{A}_h$.
6: Check stop condition.
7: **end for**

5 Numerical example

In this section, we use both two-dimensional and three-dimensional numerical experiments to validate the model and algorithm. There are two regularization parameters and a smoothing parameter in the model. Since the numerical result is not sensitive to β and ϵ if they are small enough, in all the tests we set

$$\beta = 10^{-15} \quad \text{and} \quad \epsilon = 10^{-6}.$$

The regularization parameter λ is very important to obtain a satisfactory reconstruction. We choose λ by the discrepancy principle combined with the path-following strategy. To be precise, let $\rho \in (0, 1)$ and we apply Algorithm 1 with $\lambda_n = \lambda_0 \rho^n$ and the initial guess to be the solution of the λ_{n-1}-problem.

In our simulation the noisy data g^δ are generated by

$$g^\delta(s) = \widehat{g}(s)(1 + \delta(-1 + 2\xi)) \quad \text{on } \Gamma_c,$$

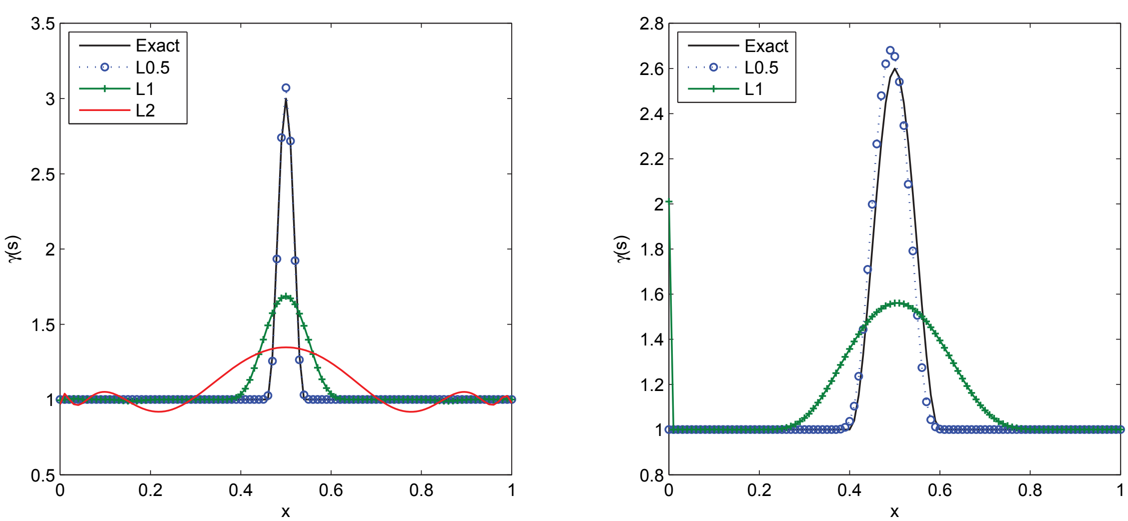
where $\widehat{g}(s)$ refers to the exact data, δ represents the noise level and ξ is uniformly random distributed on $[0, 1]$. All the computations are performed by using FreeFem++ [8] on a laptop.

Example 5.1. Let $\Omega = (0, 1) \times (0, 1)$, $\Gamma_i = (0, 1) \times \{1\}$, $\alpha = 0.5$, $f = -2$, $v = 1$ on $\{1\} \times [0, 1]$ and vanishing on the other boundaries, $u_a = 2.5$ and $\gamma^\dagger = 1$. The Robin coefficient to be recovered $\widehat{\gamma}$ is given by

$$\widehat{\gamma} = \gamma^\dagger + \left(\sin\left(\frac{\pi}{0.04}(s - 0.48)\right) + 1\right)\chi_{[0.46,0.54]}.$$

We consider here a noise free case.

Observe that the support of $\widehat{\gamma} - \gamma^\dagger$ has a very small size 0.08 which leads to an extremely ill-posed system. In Figure 1 (a), one can find the reconstruction by L^2, L^1 and $L^{0.5}$ by setting $\lambda = 1\text{e}{-7}$. The L^1- and L^2-model are solved using the algorithm in [16] and [14], respectively. Clearly only $L^{0.5}$-regularization has a satisfactory result. To get a good reconstruction by L^1-regularization one needs to choose $\lambda \le 1\text{e}{-13}$, and L^2-regularization never has a sparse reconstruction.



(a) Example 5.1. **(b)** Example 5.2.

Figure 1. The exact and numerical Robin coefficients for (a) Example 5.1 with $L^{0.5}, L^1, L^2$-regularization term and (b) for Example 5.2 with 10% noise (right panel).

Example 5.2. The problem setting is same as in Example 5.1 except for

$$\widehat{\gamma} = \gamma^\dagger + \left(\sin\left(\frac{\pi}{0.1}(s - 0.4)\right) + 1\right)\chi_{[0.4,0.6]}.$$

Let the noise level $\delta = 10\%$.

In this example the support is a bit larger (with size 0.2) but the noise is also large. Figure 1 (b) shows the reconstruction by L^1- and $L^{0.5}$-regularization. In the large noise case, $L^{0.5}$-regularization is much better than L^1-regularization.

Example 5.3. The problem setting is same as in Example 5.2. This example involves two parts. In the first part we use $q = 0.5$ but consider the different noise level $\delta = 1\%, 3\%, 10\%, 15\%$. In the second part we fix noise level $\delta = 3\%$ and choose different $q = 0.9, 0.7, 0.5, 0.3, 0.1$. The numerical results are presented in Figure 2. We can find that to reconstruct a very sparse solution, smaller q is better.

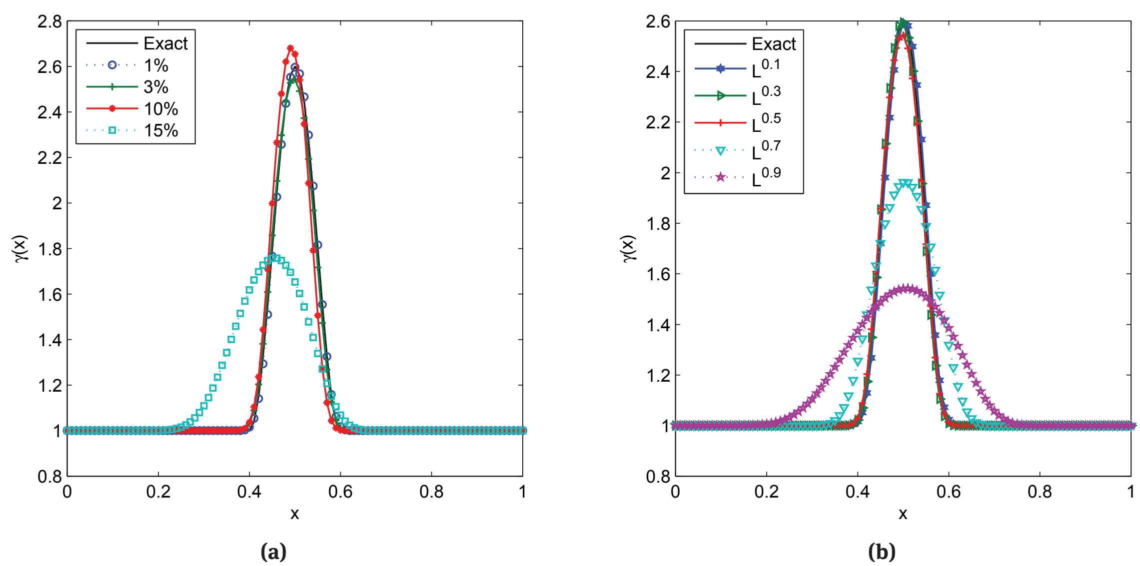


Figure 2. The exact and numerical Robin coefficients for Example 5.3 with (a) different noise level and (b) different q.

Example 5.4. Let $\Omega = (0, 1)^3$, $\Gamma_i = (0, 1) \times (0, 1) \times \{1\}$, $\alpha = 1$, $f = 6$,

$$v = \begin{cases} -2 & \text{on } x = 0 \text{ or } y = 0 \text{ or } z = 0, \\ 2 & \text{on } x = 1 \text{ or } y = 1, \end{cases}$$

$u_a = 1$, $\gamma^\dagger = 2$, $\delta = 10^{-3}$ and

$$\widehat{\gamma} = \gamma^\dagger + \sin\left(3\pi\left(x - \frac{2}{3}\right)\right)\sin\left(3\pi\left(y - \frac{2}{3}\right)\right)\chi(x)_{[\frac{1}{3},\frac{2}{3}]}\chi(y)_{[\frac{1}{3},\frac{2}{3}]}.$$

Figure 3 shows the exact and reconstruction Robin coefficient. It can be found the reconstruction is reasonable under $L^{0.5}$-regularization.

Example 5.5. Let $\Omega = (0, 1) \times (0, 1)$, $\Gamma_i = (0, 1) \times \{1\}$, $\alpha = 1$, $f = -4$, $v = 2$ on $\{1\} \times [0, 1]$ and vanishing on the other boundaries, $u_a = x^2 + y^2 + 1$ and $\gamma^\dagger = 1$. Let the noise level $\delta = 0.5\%$. The Robin coefficient to be recovered $\widehat{\gamma}$ is given by

$$\widehat{\gamma} = \gamma^\dagger + \sin\left(2\pi\left(x - \frac{1}{4}\right)\right) + 1.$$

In this example, the desired solution $\widehat{\gamma}$ is not sparse. The sparse promoted L^q-regularization may not get a satisfactory reconstruction, since we choose the regularization parameter β very small in praxis. Figure 4 (a) shows that the $L^{0.5}$- and $L^{0.8}$-regularization do not provide a good result, to compare with L^1- and L^2-regularization in Figure 4 (b).

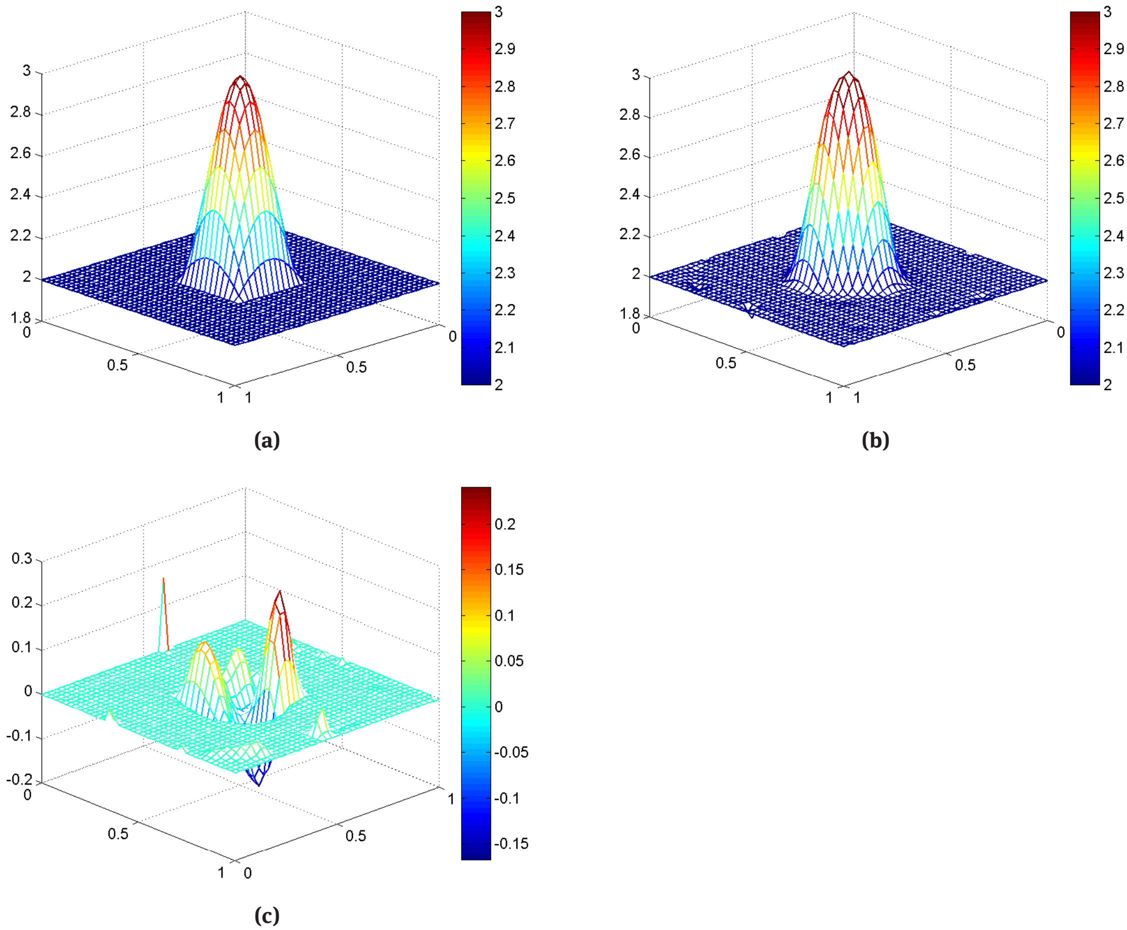


Figure 3. (a) The exact Robin coefficients for Example 5.4, (b) the numerical results with $L^{0.5}$-regularization term with 0.1% noise and (c) the error plot.

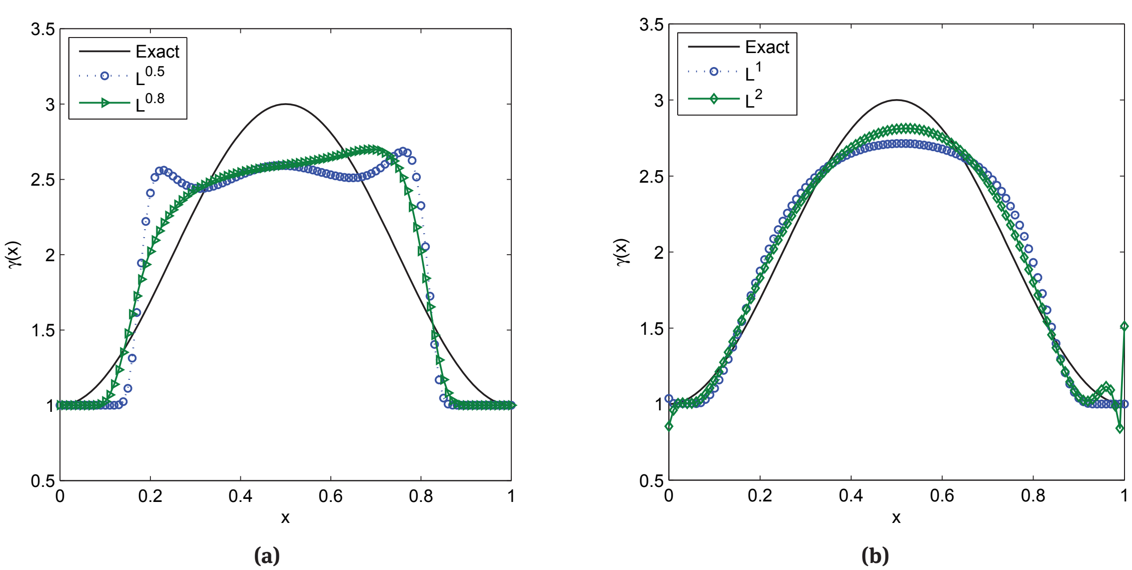


Figure 4. (a) The exact and numerical Robin coefficients for Example 5.5 with $\delta = 1\%$. $L^{0.5}$- and $L^{0.8}$-regularization and (b) L^1- and L^2-regularization.

6 Conclusion

In this paper, we considered the inverse Robin problem for an elliptic equation by adopting L^q-regularized ($q \in (0, 1)$) output least square formulation. We proved the existence of the minimizer and the convergence of the finite element discretization. A modified Newton method was adopted to solve the KKT system. Numerical simulations for the two-dimensional and three-dimensional examples validated the proposed model and algorithm.

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